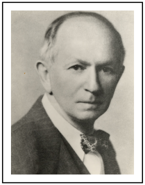
Lotka-Volterra System: Outline

View the Resource folder online.

Background: Lotka and Volterra



Alfred J Lotka[1]

Vito Volterra

Lotka *Elements of Physical Biology* (1925). [Dover reprint *Elements of Math Biology* (1956).]
"Analytical Note on Certain Rhythmic Relations in Organic Systems," *Proc Nat Acad Sci* **6** (1920), 410-415.

Volterra "Fluctuations in the abundance of a species considered mathematically," *Nature* **118** (1926), 558-560.
Translation(?) of "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Mem. R. Accad. Naz. dei Lincei* **2** (1926), 31-113.

[1] *Also see the Wikipedia entries for* Lotka *and* Voltera

Background: Beyond Logistics

Alfred Lotka was a statistician and chemist. Lotka's ideas grew from his work on oscillating reactions in chemistry. (See, e.g., Lotka, A. J., "Contribution to the Theory of Periodic Reactions," *J. Phys. Chem.*, **14**, 1910 (271-274).)

Vito Volterra was an Italian mathematician. Before World War I, he worked in differential equations, mechanics, and mathematical physics. After the war, he turned to mathematical biology and studied Verhulst's logistic model. From there, he developed his theories on population dynamics.

System of Differential Equations

Let w be the number of predators and m be the number of prey:

$$\frac{dw}{dt} = -r_w w + \alpha_w w m$$
$$\frac{dm}{dt} = +r_m m - \alpha_m w m$$

where r_w and r_m are the growth constants and α_w and α_m are interaction constants. All constants are positive.

This system doesn't have a closed form solution, so we'll have to use other methods to study it.

(See *Wolves & Moose of Isle Royale*.)

Three Forms of the System

Three useful forms of the system are:

$$\begin{cases} \dfrac{dw}{dt} = -r_w w + \alpha_w w m \\[2ex] \dfrac{dm}{dt} = +r_m m - \alpha_m w m \end{cases}$$

$$\begin{cases} \dfrac{w'}{w} = \alpha_w m - r_w \\[2ex] \dfrac{m'}{m} = -\alpha_m w + r_m \end{cases}$$

$$\begin{cases} w' = +w m \left(\alpha_w - \dfrac{r_w}{m} \right) \\[2ex] m' = +w m \left(\dfrac{r_m}{w} - \alpha_m \right) \end{cases}$$

Equilibrium and Phase Plane Analysis

Assume w and m are positive. Then

$$\left\{ \begin{array}{c} \dfrac{dw}{dt} = 0 \\ \dfrac{dm}{dt} = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} -r_w + \alpha_w m = 0 \\ +r_m - \alpha_m w = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} m = r_w/\alpha_w \\ w = r_m/\alpha_m \end{array} \right\}$$

There is an equilibrium point at $(w,m) = (r_m/\alpha_m, r_w/\alpha_w)$. The predator's parameters determine the prey equilibrium and the prey's parameters determine the predator equilibrium!

Break the first quadrant into 4 pieces bounded by the lines where $dw/dt = 0$; i.e., $w = r_m/\alpha_m$ and where $dm/dt = 0$; i.e., $m = r_w/\alpha_w$. Analyze these four regions.

Phase Plane

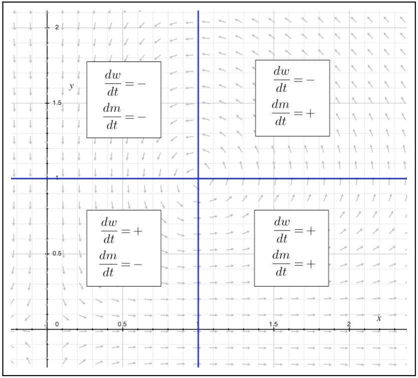
Write the DE as

$$\frac{dw}{dt} = w \cdot (-r_w + \alpha_w m)$$
$$\frac{dm}{dt} = m \cdot (r_m - \alpha_m w),$$

so that

$$\text{sign}(w') = \text{sign}(-r_w + \alpha_w m)$$

$$\text{sign}(m') = \text{sign}(+r_m - \alpha_m w)$$



Phase Plane, II

Consider dw/dm:

$$\frac{dw/dt}{dm/dt} = \frac{dw}{dm} = \frac{\alpha_w - r_w/m}{-\alpha_m + r_m/w}$$

$$\left(\frac{r_m}{w} - \alpha_m\right) dw = \left(\alpha_w - \frac{r_w}{m}\right) dm$$

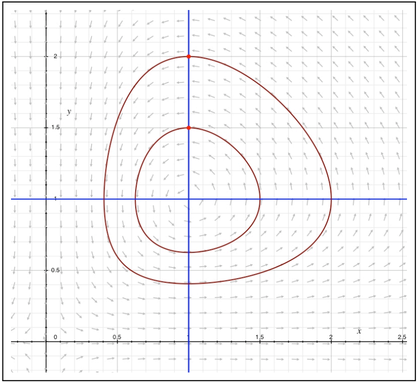
$$r_m \ln(w) - \alpha_m w = \alpha_w m - r_w \ln(m) + k$$

$$\ln(w^{r_m} m^{r_w}) = \alpha_w m + \alpha_m w + k$$

$$\frac{w^{r_m}}{e^{\alpha_m w}} \cdot \frac{m^{r_w}}{e^{\alpha_w m}} = e^k = K > 0$$

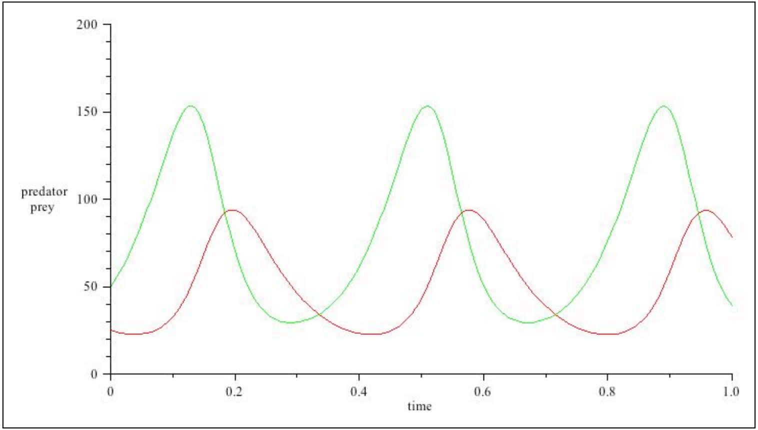
i.e., $f_1(w) \cdot f_2(m) = K$.

So phase plane curves must be closed cycles.



What does this mean?

Solution Curves



Each population cycles with the predator lagging the prey. (*Why?*)

Look at the Maple worksheet.

Exercises

1. Write a guides for producing slope fields, phase portraits, and solutions curves for Lotka-Volterra systems using either
   a. a graphing calculator,
   b. Maple,
   c. another software program of your choosing.

2. Write a report on Michael Gilpin's 1972 note in *The American Naturalist* "Do Hares Eat Lynx?" See the Rabbit of Caerbannog.

3. Analyze the *Voter Tendency model:* Let D be the number of Democrats and R be the number of Republicans. If p_{DR} percent of Democrats convert to Republicans and p_{RD} percent of the Republicans transmogrify to Democrats, then

$$\frac{dD}{dt} = -p_{DR}D + p_{RD}R$$
$$\frac{dR}{dt} = +p_{DR}D - p_{RD}R$$

Exercises, II

4. Analyze the *Competitive Hunter model:* Let S be the number of spotted owls and H be the number of hawks in a forest, then

$$\frac{dS}{dt} = r_s S - \alpha_s SH$$
$$\frac{dH}{dt} = r_h H - \alpha_h SH$$

with appropriate positive constants.

5. Analyze the *Mutualism model:* Let B be the number of bees and F be the number of avocado flowers, then

$$\frac{dB}{dt} = -r_b B + \alpha_b BF$$
$$\frac{dF}{dt} = -r_f F + \alpha_f BF$$

with appropriate positive constants.

Exercises, III

6. Report on the "Ratio-dependent" predator-prey model, (see, e.g., Eichner & Pethig).
7. Report on the *Reuters* story "Overfishing of sharks makes scallops vanish" and how population models could help us to understand the situation. (Archived article.) Investigate *trophic cascade*.
8. Report on the ecology experiment designed in Teaching Issues and Experiments in Ecology and how it uses or modifies the Lotka-Volterra model.
9. Report on the paper "Dynamics of a Lotka-Volterra type model with applications to marine phage population dynamics" by Gavin, *et al*, and how they modify the Lotka-Volterra model.
10. Report on the article "Application of quantitative models from population biology and evolutionary game theory to tumor therapeutic strategies" from *Molecular Cancer Therapeutics* and its use of Lotka-Volterra models. (Archived article.)