Formulas: Euler's Method

Initial Value Problem: $y' = f(t,y)$ and $y(t_0) = y_0$

TAYLOR'S THEOREM:	$y(t_1) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(\xi)h^2$
DIFFERENTIATING $f(t,y)$:	If $y' = f(t,y)$, then[1] $y'' = \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial y} \cdot y'$
EULER'S METHOD STEP:	$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + y'(t_n, y_n)\, h \end{cases}$
ONE-STEP ERROR:	$\varepsilon = y_k - y(t_k)$ *(error = predicted − actual)*
ONE-STEP ERROR BOUND:	If $\lvert y''\rvert \le M$, then $\lvert\varepsilon\rvert \le \frac{M}{2}h^2$
TOTAL ERROR BOUND:	If $\lvert y''\rvert \le M$ and $C = \frac{1}{2}(t_n - t_0)M$, then $\lvert\varepsilon_T\rvert \le C \cdot h = \mathscr{O}(h) = \mathscr{O}(1/n)$

[1] From $df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial y}\,dy$.

Formulas: Heun's Method

Initial Value Problem: $y' = f(t,y)$ and $y(t_0) = y_0$

HEUN'S METHOD STEP: Let $m_L = f(t_n, y_n)$ and set $y_p = y_n + m_L h$. Then find $m_R = f(t_n + h, y_p)$. The step is:

$$\begin{cases} t_{n+1} = t_n + h \\ y_{n+1} = y_n + \frac{1}{2}\left[m_L + m_R\right] h \end{cases}$$

(Heun's is a *predictor-corrector method:*
y_p ◀ predictor; m_R ◀ corrector)

ONE-STEP ERROR BOUND: If $|y''| \le M$, then $|\varepsilon| \le \frac{M}{12} h^3$

TOTAL ERROR BOUND: If $|y''| \le M$, then $|\varepsilon_T| = \mathcal{O}(h^2) = \mathcal{O}(1/n^2)$

Formulas: Maple Code

Maple Code for Solving $y' = f(t,y)$ when $y(t_0) = y_0$

```
Euler := (P,h) -> [P1 + h, P2 + f(P1,P2)·h];
Heun := (P,h) -> [P1 + h, P2 + 1/2·(f(P1,P2) + f(P1 + h, P2 + f(P1,P2)·h))·h]

DESolve := proc(t0, tn, y0, h, Method)
  local N, Pts, i;
  global f, Euler, Heun;
  N := ceil((tn - t0)/h);
  Pts := [t0, y0], Method([t0, y0], h);
  for i from 2 to N do
    Pts := Pts, Method(Pts[i], h);
    end do;
  return([Pts]);
  end proc:
```

- *Example:* First define $f(t,y) = 0.05y \cdot (1 - y/5)$. Then

```
theEulerPts := DESolve(0, 5.0, 3.0, 0.5, Euler);
theHeunPts := DESolve(0, 5.0, 3.0, 0.5, Heun);
plot([theEulerPts, theHeunPts]);
```

Formulas: Runge-Kutta Methods

Initial Value Problem: $y' = f(t,y)$ and $y(t_0) = y_0$

Runge-Kutta Method Step: Let
$$k_1 = f(t_n, y_n),$$
$$k_2 = f(t_n + \tfrac{h}{2}, y_n + k_1 \cdot \tfrac{h}{2}),$$
$$k_3 = f(t_n + \tfrac{h}{2}, y_n + k_2 \cdot \tfrac{h}{2}),$$
$$k_4 = f(t_n + h, y_n + k_3 \cdot h).$$

Then
$$y_{n+1} = y_n + \tfrac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4) \cdot h$$

One-Step Error Bound: If $|y^{(4)}| \le M$, then $|\varepsilon| \le C \cdot h^5$

Total Error Bound: If $|y^{(4)}| \le M$, then $|\varepsilon_T| = \mathcal{O}(h^4) = \mathcal{O}(1/n^4)$

Formulas: RK4 Maple Code

RK4 Maple Code for Solving $y' = f(t,y)$

```
RK4 := proc(P, h)
  local k, Q;
  k_1 := f(P_1, P_2);
  k_2 := f(P_1 + h/2, P_2 + k_1 · h/2);
  k_3 := f(P_1 + h/2, P_2 + k_2 · h/2);
  k_4 := f(P_1 + h, P_2 + k_3 · h);
  Q := [P_1 + h, P_2 + 1/6 (k_1 + 2 k_2 + 2 k_3 + k_4) · h];
  return(Q);
  end proc:
```

- *Example:* First define $f(t,y) = 0.05\,y \cdot (1 - y/5)$. Then find $y(5)$ when $y_0 = 3$:

```
t0, tn, y0, h := 0, 5.0, 3.0, 0.5;
theRK4Pts := DESolve(t0, tn, y0, h, RK4);
plot([theRK4Pts]);
```

Comparison

Comparing the Methods

Method	**Order**	**Basis**	**Cost**
Euler's method	$\mathcal{O}(1/n)$	Left endpoint integration	Low
Heun's method	$\mathcal{O}(1/n^2)$	Trapezoid rule	Medium
Runge-Kutta method[2]	$\mathcal{O}(1/n^4)$	Simpson's rule	High

Test Case

Apply each of the methods to the initial value problem

$$y' = 9y \cdot (1-y), \quad y_0 = 1.25$$

to calculate $y(10)$ beginning with $h = 0.29$.

[2] If you're curious about the derivation of RK methods, see, e.g., *Runge-Kutta Methods*

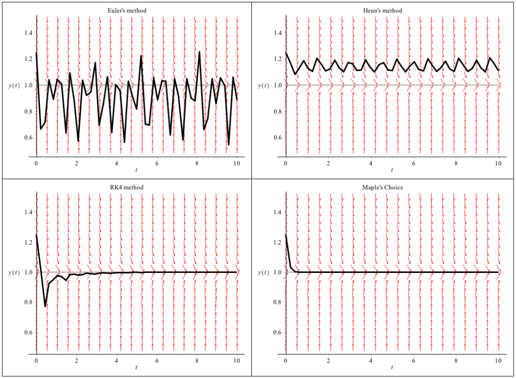
Comparison Spoiler

$de := y'(t) = 9 \cdot y(t) \cdot (1 - y(t))$:

$Args := de, [y], t = 0..10, y = 0.5..1.5, [[y(0) = 1.0], [y(0) = 1.25]], stepsize = 0.29, linecolor = [grey, black]$:

$thePlots := array($
$[[DEtools[DEplot](Args, method = classical[foreuler], title = $ "Euler's method" $),$
$DEtools[DEplot](Args, method = classical[heunform], title = $ "Heun's method" $)],$
$[DEtools[DEplot](Args, method = classical[rk4], title = $ "RK4 method" $),$
$DEtools[DEplot](Args, title = $ "Maple's Choice" $)]])$:

$plots[display](thePlots)$



Formulas: Systems

Using Numerical Methods for Systems

Simply apply the method using all variables in the step formula:

$$\begin{cases} x'(t) = f(t,x,y) \\ y'(t) = g(t,x,y) \end{cases} \quad \text{with} \quad \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

is approximated with Euler's method by

$$\begin{cases} t_{n+1} = t_n + h \\ x_{n+1} = x_n + f(t_n, x_n, y_n) \cdot h \\ y_{n+1} = y_n + g(t_n, x_n, y_n) \cdot h \end{cases}$$

We can use the Maple function

$\texttt{EulerSystem} := [P, h] \to [P_1 + h,\ P_2 + f(P_1, P_2, P_3) \cdot h,\ P_3 + g(P_1, P_2, P_3) \cdot h];$

NB: Numerical methods can be applied to a higher order DE by converting it to a system:

$$y'' + a(t)y' + b(t)y = f(t) \implies \begin{cases} x = y' \\ x' = f(t) - a(t)x - b(t)y \end{cases}$$