Partial Fractions Paths to Shortcuts

1. An Example with the Standard Algebra

Let

$$f(x) = \frac{x^2+1}{x^3-5x^2+8x-4}.$$

The roots of $q(x)$, the denominator, are 1 and 2^*. Maple's *partial fraction decomposition*† of f is

$$f(x) = \frac{2}{x-1} + \frac{-1}{x-2} + \frac{5}{(x-2)^2} \qquad \text{giving} \qquad \int f(x)\,dx = 2\ln(x-1) - \ln(x-2) - \frac{5}{x-2}.$$

1. The denominator factors as $(x-1)\cdot(x-2)^2$, so the form of our partial fraction decomposition of $f(x)$ is

$$\frac{x^2+1}{x^3-5x^2+8x-4} = \frac{A}{x-1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2}$$

2. Multiply both sides by the denominator to clear fractions. Collect like terms to find

$$\begin{aligned} x^2+1 &= A(x-2)^2 + B_1(x-1)(x-2) + B_2(x-1) \\ &= (A+B_1)x^2 + (-4A-3B_1+B_2)x + (4A+2B_1-B_2) \end{aligned}$$

3. Generate a system of equations by matching the coefficients on the right with those on the left.

$$\begin{aligned} A + \ B_1 \qquad &= 1 && \text{(coefficients of } x^2\text{)} \\ -4A - 3B_1 + B_2 &= 0 && \text{(coefficients of } x\text{)} \\ 4A + 2B_1 - B_2 &= 1 && \text{(the constants)} \end{aligned}$$

Solve the system using your favourite method to see $A = 2$, $B_1 = -1$, and $B_2 = 5$.

Whence

$$\frac{x^2+1}{x^3-5x^2+8x-4} = \frac{2}{x-1} + \frac{-1}{x-2} + \frac{5}{(x-2)^2}$$

Now integration is easy.

$$\begin{aligned} \int \frac{x^2+1}{x^3-5x^2+8x-4}\,dx &= 2\ln(x-1) - \ln(x-2) - \frac{5}{x-2} + C \\ &= \ln\left(\frac{(x-1)^2}{x-2}\right) - \frac{5}{x-2} + C \end{aligned}$$

And so are Laplace transforms.

$$\begin{aligned} \mathscr{L}^{-1}\left[\frac{s^2+1}{s^3-5s^2+8s-4}\right](t) &= \mathscr{L}^{-1}\left[\frac{2}{s-1}\right](t) + \mathscr{L}^{-1}\left[\frac{-1}{s-2}\right](t) + \mathscr{L}^{-1}\left[\frac{5}{(s-2)^2}\right](t) \\ &= 2e^t - e^{2t} + 5te^{2t} \end{aligned}$$

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* Recall the Rational Root Theorem:

Suppose $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has a factor $(x-r)$ for some rational number r. Then $r = \pm$(factor of a_0)/(factor of a_n).

† From: `convert(f(x),parfrac)`.

2. An Example with Clever Algebra

Again, let

$$f(x) = \frac{x^2+1}{x^3-5x^2+8x-4} = \frac{A}{x-1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2}.$$

The roots of $q(x)$, the denominator, are still 1 and 2.

Consider the following computations.

1. For $x = 1$: Look at $f_1(x) = (x-1) \times f(x)$.

$$f_1(x) = \frac{(x-1)(x^2+1)}{x^3-5x^2+8x-4} = \frac{(x-1)(x^2+1)}{(x-1)(x-2)^2}$$
$$= \frac{x^2+1}{(x-2)^2}$$
$$f_1(1) \to \boxed{2} = A \qquad \textit{(Equivalent to } \lim_{x\to 1}[(x-1)f(x)].\text{)}$$

2. For $x = 2$. Look at $f_2(x) = (x-2) \times f(x)$.

$$f_2(x) = \frac{(x-2)(x^2+1)}{x^3-5x^2+8x-4} = \frac{(x-2)(x^2+1)}{(x-1)(x-2)^2}$$
$$= \frac{x^2+1}{(x-1)(x-2)}$$
$$f_2(2) \to \frac{5}{0} \quad \textit{undefined!}$$

   This technique can't find the coefficient in $B_1/(x-2)$.

3. Look at $f_3(x) = (x-2)^2 \times f(x)$.

$$f_3(x) = \frac{(x-2)^2(x^2+1)}{x^3-5x^2+8x-4} = \frac{(x-2)^2(x^2+1)}{(x-1)(x-2)^2}$$
$$= \frac{x^2+1}{(x-1)}$$
$$f_2(2) \to \boxed{5} = B_2$$

In words:

- *Factor the denominator.*
- *Successively multiply both sides by each factor, reducing the fraction each time.*
- *Set x to the value that makes the factor equal to zero.*
- *Evaluate to find the partial fraction coefficient for that factor.*
- *Use standard algebra (systems of linear equations) to find the coefficients for any 'lower power terms'.*

3. Generically with Calculus

Suppose $f(x) = p(x)/q(x)$ where p and q are polynomials with no common factors. Let r be a root of q, and let $(x-r)^n$ be the highest power of $(x-r)$ that divides q; i.e., r is a root of q with multiplicity n.

Suppose the highest power n for some root r is 1; that is, suppose r is a *simple root*. Then

$$\begin{aligned}(x-r)^1 f(x) &= (x-r)^1 \frac{p(x)}{q(x)} \\ &= \frac{p(x)}{\frac{q(x)}{x-r}} = \frac{p(x)}{\frac{q(x)-q(r)}{x-r}} \qquad \text{(since } q(r)=0\text{)}\end{aligned}$$

Take the limit of both sides as $x \to r$.

$$\lim_{x\to r}(x-r)f(x) = \lim_{x\to r}\frac{p(x)}{\frac{q(x)-q(r)}{x-r}} = \frac{\lim\limits_{x\to r} p(x)}{\lim\limits_{x\to r}\frac{q(x)-q(r)}{x-r}} = \frac{p(r)}{q'(r)}$$

Revisit the previous example where

$$p(x) = x^2+1 \quad \text{and} \quad q(x) = x^3-5x^2+8x-4 = (x-1)(x-2)^2.$$

So

$$p(x) = x^2+1 \quad \text{and} \quad q'(x) = 3x^2-10x+8,\ q''(x) = 6x-10$$

1. Then
$$\frac{p(1)}{q'(1)} = \frac{2}{1} = \boxed{2} = A$$
2. Try
$$\frac{p(2)}{q'(2)} = \frac{5}{0} \quad \textit{undefined}$$
3. Try
$$\frac{p(2)}{q''(2)/2!} = \frac{5}{2/2} = \boxed{5} = B_2$$

In words:

- *Find the roots of the denominator and their multiplicities (powers).*
- *Differentiate the denominator as needed.*
- *For a root r of multiplicity n, compute the partial fraction coefficient for $\frac{A}{(x-r)^n}$ with $A = n! \times \frac{p(r)}{q^{(n)}(r)}$.*
- *Use algebra to find the coefficients for 'lower power terms.'*

It would take a *lot* of work, but we could show that the coefficient in $\frac{A_i}{(x-r)^i}$ is given by

$$A_i = \frac{h^{(n-i)}(r)}{(n-i)!} \qquad \text{for } i = 1..n$$

where $h(x) = \left[(x-r)^n \times \frac{p(x)}{q(x)}\right]$ and $h^{(0)}(x) = h(x)$. So B_1 from above would be $B_1 = h'(2)/1! = -1$.