Composite Simpson's Rule

Suppose $x_0 = a$, $x_n = b$, and $h = (b-a)/n$ where n is even. Choose $x_k = a + kh$. Then

$$S(f,P) = \frac{h}{3}\left[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + \cdots + 2f(a+(n-2)h) + 4f(a+(n-1)h) + f(b)\right] \tag{1}$$

$$= \frac{h}{3}\sum_{i=1}^{n/2}\left[f(a+2(i-1)h) + 4f(a+(2i-1)h) + f(a+2ih)\right] \tag{2}$$

Theorem 1 (Composite Simpson's Rule Error Bound)**.** *If $f^{(4)}(x)$ is continuous on $[a,b]$, then*

$$\left|\int_a^b f(t)\,dt - S(f,n)\right| = -\frac{1}{90}\cdot f^{(4)}(\xi)\cdot h^5 + O(h^6) \tag{3}$$

$$= \frac{b-a}{180}\cdot f^{(4)}(\xi)\cdot h^4 + O(h^6)$$

for some $\xi \in (a,b)$.

Problem

1. Consider $F_s(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\,dt$. Determine the number n that ensures an error $< 10^{-5}$ for calculating $F_s(1)$.

A Recursive Error Estimate

Set

$$\mathscr{I} = \int_a^b f(x)\,dx = S(a,b) + E(a,b) = S^{(1)} + E^{(1)} \tag{4}$$

where $S^{(1)}(a,b) = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$ and $E^{(1)}(a,b) = -\frac{1}{90}\,(h)^5\,C$ assuming $f^{(4)}(x) = C$ is constant on $[a,b]$. Bisect the interval to have $[a,b] = [a,c] + [c,b]$, then

$$\mathscr{I} = S^{(2)} + E^{(2)} \tag{5}$$

where $S^{(2)} = S(a,c) + S(c,b)$, and

$$E^{(2)} = -\frac{1}{90}\left(\frac{h}{2}\right)^5 C - \frac{1}{90}\left(\frac{h}{2}\right)^5 C$$

$$= \frac{1}{16}\left[-\frac{1}{90}\,(h)^5\,C\right] = \frac{1}{16}E^{(1)} \tag{6}$$

So $16E^{(2)} = E^{(1)}$. Now, subtract (5) from (4) to obtain

$$S^{(2)} - S^{(1)} = E^{(1)} - E^{(2)} = 15E^{(2)}$$

Use the formula above to replace $E^{(2)}$ in (5):

$$\mathscr{I} = S^{(2)} + E^{(2)} = S^{(2)} + \frac{1}{15}\left[S^{(2)} - S^{(1)}\right]$$

Thus

$$\frac{1}{15}\left|S^{(2)} - S^{(1)}\right| < \varepsilon \tag{7}$$

Our criterion will be: Compute $S^{(1)}$ and $S^{(2)}$. Check (7). If the error tolerance is satisfied, quit. If not, bisect the two intervals and repeat the procedure.

Adaptive Composite Simpson's Rule Integration

Set $h = (b-a)/2$ and the desired error tolerance to $\varepsilon > 0$. Then calculate

$$S_1 = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b)\right]$$

$$S_2 = \frac{h}{6}\left[f(a) + 4f(a+h/2) + 2f(a+h) + 4f(a+3h/2) + f(b)\right]$$

Decision Rule: IF $|S_1 - S_2| < 15\varepsilon$,

THEN *return* $\left[\int_a^b f(x)\,dx \approx \frac{16S_2 - S_1}{15} \text{ to within } \varepsilon\right]$,

ELSE *bisect* $[a,b] \to [a,c] + [c,b]$ *with* $c = (a+b)/2$ *and repeat with each subinterval.*

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recursive real function Simpson(f, a, b, ε, level, level_max)
      result(simpson_result)
integer level, level_max;       real a, b, c, d, e, h
external function f
level ← level + 1
h ← b − a
c ← (a + b)/2
one_simpson ← h[f(a) + 4f(c) + f(b)]/6
d ← (a + c)/2
e ← (c + b)/2
two_simpson ← h[f(a) + 4f(d) + 2f(c) + 4f(e) + f(b)]/12
if level ≧ level_max then
      simpson_result ← two_simpson
      output   "maximum level reached"
else
      if |two_simpson − one_simpson| < 15ε then
            simpson_result ← two_simpson + (two_simpson − one_simpson)/15
      else
            left_simpson ← Simpson(f, a, c, ε/2, level, level_max)
            right_simpson ← Simpson(f, c, b, ε/2, level, level_max)
            simpson_result ← left_simpson + right_simpson
      end if
end if
end function Simpson
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Cheney & Kincaid, pg 224.

Note: *Cheney & Kincaid* use $h = b - a$.