Linear Programming

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Outerlude

Today

Lego Furniture Industries, Inc.

LFI, Inc., Production Problem

LFI, Inc., makes tables and chairs (*competing activities*) using a limited amount of large and small Legos (*limited resources*). Profit for each table is $\$100$ while profit for each chair is $\$45$. We need to identify the number of tables and chairs to produce to maximize profit while not using more Legos than are available.

Decision Variables: Factors controlled by the decision maker:

$$x_1 = \text{ the number of tables produced per day}$$
$$x_2 = \text{ the number of chairs produced per day}$$

Objective function: The objective is to maximize profit P

$$P = 100x_1 + 45x_2$$

Constraints: Restrictions limiting availability of resources.

1. It takes 1 large & 2 medium Legos to produce a table; 1 medium & 2 smalls to produce a chair.
2. We have 6 large, 16 medium, & 10 small Legos available each day.

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Definition

Linear Programming is a method for optimizing a linear objective function given a set of linear constraints.

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$$\text{maximize:} \quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\text{subject to:} \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \end{cases}$$

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or in matrix notation

$$\text{max: } z = \vec{c} \cdot \vec{x}$$

$$\text{subject to: } \mathbf{A}\vec{x} \le \vec{b}$$

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LP Eg 1

Example

Determine the minimum price formula of feed for laboratory rabbits. A rabbit requires at least 24 g fat, 36 g carbohydrates, and 4 g protein, but receives no more than 5 oz of feed per day.

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Brand	**Protein** (g/oz)	**Fat** (g/oz)	**Carb.** (g/oz)	**Price** ($/oz)
Crunchies	2	8	12	$0.30
Nuggets	1	12	12	$0.25
Required	4 g	24 g	36 g	

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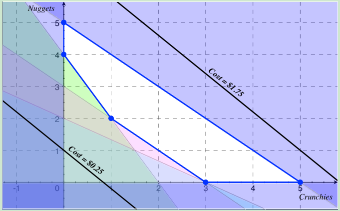
Nuggets	1	12	12	$0.25
Required	4 g	24 g	36 g	

$$\text{Cost} = 0.30c + 0.25n$$

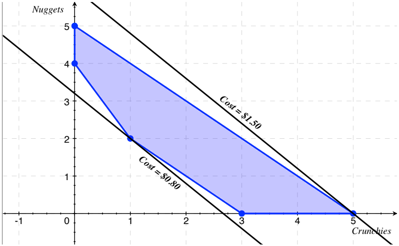
$$\begin{cases} 2c + 1n \geq 4 \\ 8c + 12n \geq 24 \\ 12c + 12n \geq 36 \\ c + n \leq 5 \\ c, n \geq 0 \end{cases}$$



LP Eg 1, Graphical Solution

Evaluate the cost function at each of the vertices of the *feasible region*.

Crunchies	*Nuggets*	**Cost**
3 oz	0 oz	\$0.90
5 oz	0 oz	\$1.50
0 oz	5 oz	\$1.25
0 oz	4 oz	\$1.00
1 oz	2 oz	\$0.80



From the chart, we see that the minimum cost of \$0.80 results from feeding 1 oz of *Crunchies* mixed with 3 oz of *Nuggets*.

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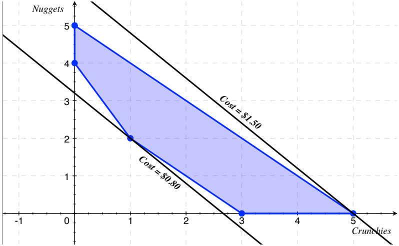
0 oz	4 oz	\$1.00
1 oz	2 oz	\$0.80 ◀



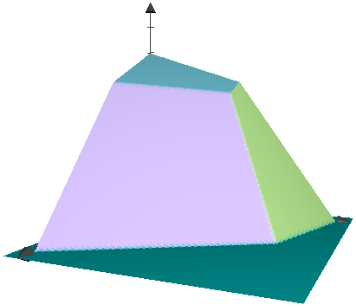
From the chart, we see that the minimum cost of \$0.80 results from feeding 1 oz of *Crunchies* mixed with 3 oz of *Nuggets*.

LP Eg 2: 3D

Example

Maximize $P = 5x + 3y + 3z$ subject to

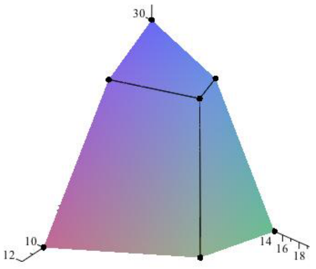
$$75x + 150y + 45z \leq 2250$$
$$120x + 60y + 80z \leq 2400$$
$$100x + 40y + 25z \leq 1000$$
$$x, y, z \geq 0$$



LP Eg 2: 3D, Solution

Example (Solution)

Vertex	*Objective*
$[0,0,0]$	0.00
$[10,0,0]$	50.00
$[4,0,24]$	92.00
$[0,0,30]$	90.00
$[0,7.7,24.2]$	95.81
$[0,15,0]$	45.00
$[5,12.5,0]$	62.50
$[1.4,7.7,22.2]$	96.44



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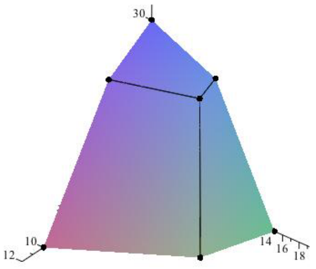

Max $P = 96.44$ at vertex $[1.4, 7.7, 22.2]$.

(min $P = 0$ at $[0,0,0]$.)

"Big" LP

Project

TriCity PowerCo *has three power plants supplying the electric needs of four cities. The associated supply of each plant and demand of each city is given in the table. The cost of sending 1 million kwh of electricity from a generating plant to a city depends on the distance.*

		To				Supply (M kwh)
		City 1	City 2	City 3	City 4	
From	Plant 1	$8	$6	$10	$9	35
	Plant 2	$9	$12	$13	$7	50
	Plant 3	$14	$9	$16	$5	40
Demand (M kwh)		45	20	30	30	

There are: 12 *decision variables and* 7 *constraints (*3 *supply &* 4 *demand)!*

"Big" LP Vertices

How Many Potential Vertices?

2-D: We had $4+2$ constraints in 2 dimensions (variables) for $\binom{6}{2} = 15$ potential vertices; only 5 were corner points.

3-D: We had $3+3$ constraints in 3 dimensions for $\binom{6}{3} = 20$ potential vertices; only 8 were corner points.

12-D: We had $7+12$ constraints in 12 dimensions for $\binom{19}{12}$ $= 50{,}388$ potential vertices. How many corner points?

n-D: We have $c+n$ constraints in n dimensions for $\binom{c+n}{n} = \frac{(n+c)!}{n!\,c!}$ potential vertices.

"Big" LP Solution

x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	1	−1020

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x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	1	−1020
0			0		0		0	0		0			

"Big" LP Solution

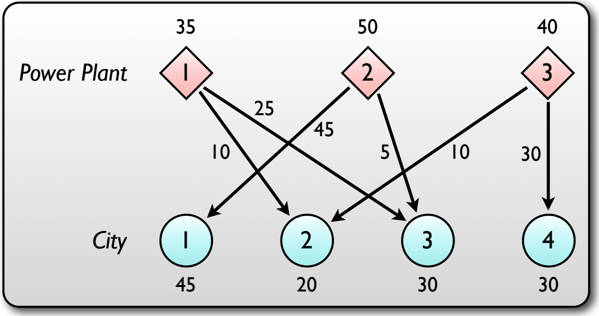
x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	1	−1020

0			0		0		0	0		0		
	10	25		45		5			10		30	1020
x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	*cost*

"Big" LP Solution Chart

The Solution

0	10	25	0	45	0	5	0	0	10	0	30	1020
x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	$cost$



The Simplex Method

Maximize $p = 5x + 3y + 3z$
subject to

$$\begin{cases} 75x + 150y + 45z & \leq 2250 \\ 120x + 60y + 80z & \leq 2400 \\ 100x + 40y + 25z & \leq 1000 \\ x, y, z & \geq 0 \end{cases}$$

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Maximize $p = 5x + 3y + 3z$ $\Longrightarrow$ $p - 5x - 3y - 3z = 0$
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s_1, s_2, and s_3 are *slack variables* making the inequalities into equations.

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s_1, s_2, and s_3 are *slack variables* making the inequalities into equations.

Tableau 1

Variable	x	y	z	s_1	s_2	s_3	p	*Const*
Ineq 1	75	150	45	1	0	0	0	2250
Ineq 2	120	60	80	0	1	0	0	2400
Ineq 3	100	40	25	0	0	1	0	1000
Obj Fcn	-5	-3	-3	0	0	0	1	0

The Simplex Method: First Pivot

Tableau 1

x	y	z	s_1	s_2	s_3	p	
75	150	45	1	0	0	0	2250
120	60	80	0	1	0	0	2400
100	40	25	0	0	1	0	1000 [2]
-5 [1]	-3	-3	0	0	0	1	0

[1] *Most negative value in the objective fcn row.*
[2] *Minimum ratio in the constant column.*

The Simplex Method: First Pivot

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x	y	z	s_1	s_2	s_3	p	
75	150	45	1	0	0	0	2250
120	60	80	0	1	0	0	2400
100	40	25	0	0	1	0	1000 [2]
-5 [1]	-3	-3	0	0	0	1	0

Tableau 2

x	y	z	s_1	s_2	s_3	p	
0	120	35.25	1	0	-0.75	0	1500
0	12	50.00	0	1	-1.20	0	1200
1	0.4	0.25	0	0	0.01	0	10
0	-1	-1.75	0	0	0.05	1	50

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[2] *Minimum ratio in the constant column.*

The Simplex Method: Second Pivot

Tableau 2

x	y	z	s_1	s_2	s_3	p	
0	120	35.25	1	0	-0.75	0	1500
0	12	50.00	0	1	-1.20	0	1200
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1	0.4	0.25	0	0	0.01	0	10
0	-1	-1.75	0	0	0.05	1	50

Tableau 3

x	y	z	s_1	s_2	s_3	p	
0	111.5	0	1	-0.705	0.096	0	654
0	0.24	1	0	0.02	-0.024	0	24
1	0.34	0	0	-0.005	0.016	0	4
0	-0.58	0	0	0.035	0.008	1	92

The Simplex Method: Third Pivot

Tableau 3

x	y	z	s_1	s_2	s_3	p	
0	111.5	0	1	-0.705	0.096	0	654
0	0.24	1	0	0.02	-0.024	0	24
1	0.34	0	0	-0.005	0.016	0	4
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0	-0.58	0	0	0.035	0.008	1	92

Tableau 4

x	y	z	s_1	s_2	s_3	p	
0	1	0	0.009	-0.006	-0.001	0	5.86
0	0	1	-0.002	0.022	-0.024	0	22.6
1	0	0	-0.003	-0.003	0.016	0	2.01
0	0	0	0.005	0.031	0.008	1	95.4

The Simplex Method: The Solution

Tableau 4

x	y	z	s_1	s_2	s_3	p	
0	1	0	0.009	-0.006	-0.001	0	5.86
0	0	1	-0.002	0.022	-0.024	0	22.6
1	0	0	-0.003	-0.003	0.016	0	2.01
0	0	0	0.005	0.031	0.008	1	95.4

The Simplex Method: The Solution

Tableau 4

x	y	z	s_1	s_2	s_3	p	
0	1	0	0.009	-0.006	-0.001	0	5.86
0	0	1	-0.002	0.022	-0.024	0	22.6
1	0	0	-0.003	-0.003	0.016	0	2.01
0	0	0	0.005	0.031	0.008	1	95.4

There is no negative value in the objective function row, so there is no pivot element: 95.4 is the optimal value of p.

Set s_1, s_2, and $s_3 = 0$. Then

$$x = 2.01, y = 5.86, \text{ and } z = 22.6 \text{ yielding maximum } p = 95.4$$

The Simplex Method: The LFI, Inc., Initial Tableau

Let $(x_1, x_2) = (\#$ of tables, $\#$ of chairs$)$. Then

$$p = \$100x_1 + \$45x_2$$

$$\begin{cases} x_1 + + s_1 = 6 \\ 2x_1 + x_2 + s_2 = 16 \\ + 2x_2 + s_3 = 10 \end{cases}$$

The Simplex Method: The LFI, Inc., Initial Tableau

Let $(x_1, x_2) = (\#$ of tables, $\#$ of chairs$)$. Then

$$p = \$100x_1 + \$45x_2$$

$$\begin{cases} x_1 + \quad\quad + s_1 \quad\quad\quad\quad = 6 \\ 2x_1 + \ x_2 \quad\quad\quad + s_2 \quad\quad = 16 \\ \quad + 2x_2 \quad\quad\quad\quad\quad + s_3 = 10 \end{cases}$$

	Tables x_1	*Chairs* x_2	s_1	s_2	s_3	*Profit* p	*const*	*Basic Variable*
Large	1	0	1	0	0	0	6	s_1
Medium	2	1	0	1	0	0	16	s_2
Small	0	2	0	0	1	0	10	s_3
Obj fcn	-100	-45	0	0	0	1	0	

The Simplex Method: The LFI, Inc., Initial Tableau

Let $(x_1, x_2) = (\#$ of tables, $\#$ of chairs$)$. Then

$$p = \$100x_1 + \$45x_2$$

$$\begin{cases} x_1 + \qquad + s_1 \qquad\qquad = 6 \\ 2x_1 + x_2 \qquad + s_2 \qquad = 16 \\ \quad + 2x_2 \qquad\qquad + s_3 = 10 \end{cases}$$

	Tables x_1	*Chairs* x_2	s_1	s_2	s_3	*Profit* p	*const*	*Basic Variable*
Large	1	0	1	0	0	0	6	s_1
Medium	2	1	0	1	0	0	16	s_2
Small	0	2	0	0	1	0	10	s_3
Obj fcn	-100	-45	0	0	0	1	0	

LFI Maple worksheet

The Simplex Method: The LFI, Inc., Final Tableau

	Tables	Chairs				Profit		Basic
	x_1	x_2	s_1	s_2	s_3	p	const	Variable
Large	1	0	1	0	0	0	6	x_1
Medium	0	1	-2	1	0	0	4	x_2
Small	0	0	4	-2	1	0	2	s_3
Obj fcn	0	0	10	45	0	1	780	

The Simplex Method: The LFI, Inc., Final Tableau

	Tables x_1	*Chairs* x_2	s_1	s_2	s_3	*Profit* p	*const*	*Basic Variable*
Large	1	0	1	0	0	0	6	x_1
Medium	0	1	-2	1	0	0	4	x_2
Small	0	0	4	-2	1	0	2	s_3
Obj fcn	0	0	10	45	0	1	780	

Build $x_1 = 6$ tables and $x_2 = 4$ chairs for maximum profit of $p = \$780$.

The Simplex Method: The LFI, Inc., Final Tableau

	Tables x_1	*Chairs* x_2	s_1	s_2	s_3	*Profit* p	*const*	*Basic Variable*
Large	1	0	1	0	0	0	6	x_1
Medium	0	1	-2	1	0	0	4	x_2
Small	0	0	4	-2	1	0	2	s_3
Obj fcn	0	0	10	45	0	1	780	

Build $x_1 = 6$ tables and $x_2 = 4$ chairs for maximum profit of $p = \$780$.

Shadow prices are

Large: 10

Medium: 45

Small: 0

The Simplex Method: The LFI, Inc., Final Tableau

	Tables x_1	Chairs x_2	s_1	s_2	s_3	Profit p	const	Basic Variable
Large	1	0	1	0	0	0	6	x_1
Medium	0	1	-2	1	0	0	4	x_2
Small	0	0	4	-2	1	0	2	s_3
Obj fcn	0	0	10	45	0	1	780	

Build $x_1 = 6$ tables and $x_2 = 4$ chairs for maximum profit of $p = \$780$.

Shadow prices are

Large: 10 *Adding one additional Large increases profit by $10.*

Medium: 45 *Adding one additional Medium increases profit by $45.*

Small: 0 *Adding one additional Small doesn't affect profit.*

LP: Resource Allocation

Problem (Allocating Resources)

A developer has 60 acres for a new subdivision of townhouses, single-story houses, and two-story houses. On one acre he can put 6 townhouses, 4 single-story houses, or 2 two-story houses. It costs \$40,000 to build a townhouse, \$50,000 for a single-story house, and \$60,000 for a two-story house. He makes a profit of \$15,000 on a townhouse, \$18,000 on a single-story house, and \$20,000 on a two-story house. He has \$2,880,000 of capital available. Townhouses require 2,500 hours of labor, single-story houses require 3,000 hours of labor, and two-story houses require 4,000 hours of labor. He has 240,000 hours of labor available.

How many houses of each type should he build to maximize profit?

The Simplex Method: The Initial Tableau

Let $(x_1, x_2, x_3) = $ (Townhouse, One story, Two story). Then

$$p = 15x_1 + 18x_2 + 20x_3$$

$$\begin{cases} \frac{1}{6}x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 + s_1 = 60 \\ 40x_1 + 50x_2 + 60x_3 + s_2 = 2880 \\ 25x_1 + 30x_2 + 40x_3 + s_3 = 2400 \end{cases}$$

The Simplex Method: The Initial Tableau

Let $(x_1, x_2, x_3) =$ (Townhouse, One story, Two story). Then

$$p = 15x_1 + 18x_2 + 20x_3$$

$$\begin{cases} \frac{1}{6}x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 + s_1 = 60 \\ 40x_1 + 50x_2 + 60x_3 + s_2 = 2880 \\ 25x_1 + 30x_2 + 40x_3 + s_3 = 2400 \end{cases}$$

	Town x_1	*Single* x_2	*Two* x_3	s_1	s_2	s_3	*Profit* p	*const*
acres	1/6	1/4	1/2	1	0	0	0	60
cost	40	50	60	0	1	0	0	2880
labor	25	30	40	0	0	1	0	2400
obj fcn	-15	-18	-20	0	0	0	1	0

The Simplex Method: The Initial Tableau

Let $(x_1, x_2, x_3) =$ (Townhouse, One story, Two story). Then

$$p = 15x_1 + 18x_2 + 20x_3$$

$$\begin{cases} \frac{1}{6}x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 + s_1 = 60 \\ 40x_1 + 50x_2 + 60x_3 + s_2 = 2880 \\ 25x_1 + 30x_2 + 40x_3 + s_3 = 2400 \end{cases}$$

	Town x_1	*Single* x_2	*Two* x_3	s_1	s_2	s_3	*Profit* p	*const*
acres	1/6	1/4	1/2	1	0	0	0	60
cost	40	50	60	0	1	0	0	2880
labor	25	30	40	0	0	1	0	2400
obj fcn	-15	-18	-20	0	0	0	1	0

Maple worksheet

The Simplex Method: The Final Tableau

	Town	Single	Two	(Acres)	(Cost)	(Labour)	Profit	
	x_1	x_2	x_3	s_1	s_2	s_3	p	const
acres	0	$1/24$	$1/4$	1	$-1/240$	0	0	48
cost	1	$5/4$	$3/2$	0	$1/40$	0	0	72
labor	0	$-5/4$	$5/2$	0	$-5/8$	1	0	600
obj fcn	0	$3/4$	$5/2$	0	$3/8$	0	1	1080

The Simplex Method: The Final Tableau

	Town x_1	*Single* x_2	*Two* x_3	*(Acres)* s_1	*(Cost)* s_2	*(Labour)* s_3	*Profit* p	*const*
acres	0	1/24	1/4	1	−1/240	0	0	48
cost	1	5/4	3/2	0	1/40	0	0	72
labor	0	−5/4	5/2	0	−5/8	1	0	600
obj fcn	0	3/4	5/2	0	3/8	0	1	1080

- 72 Townhouses $+ 0$ Single story $+ 0$ Two story $= \$1{,}080{,}000$ profit.

The Simplex Method: The Final Tableau

	Town x_1	Single x_2	Two x_3	(Acres) s_1	(Cost) s_2	(Labour) s_3	Profit p	const
acres	0	1/24	1/4	1	−1/240	0	0	48
cost	1	5/4	3/2	0	1/40	0	0	72
labor	0	−5/4	5/2	0	−5/8	1	0	600
obj fcn	0	3/4	5/2	0	3/8	0	1	1080

- 72 Townhouses $+ 0$ Single story $+ 0$ Two story $= \$1{,}080{,}000$ profit.
- Shadow prices: acres $= 0$; cost $= 3/8$; labour $= 0$.

The Simplex Method: The Final Tableau

	Town x_1	*Single* x_2	*Two* x_3	*(Acres)* s_1	*(Cost)* s_2	*(Labour)* s_3	*Profit* p	*const*
acres	0	1/24	1/4	1	−1/240	0	0	48
cost	1	5/4	3/2	0	1/40	0	0	72
labor	0	−5/4	5/2	0	−5/8	1	0	600
obj fcn	0	3/4	5/2	0	3/8	0	1	1080

- 72 Townhouses $+$ 0 Single story $+$ 0 Two story $= \$1{,}080{,}000$ profit.
- Shadow prices: acres $= 0$; cost $= 3/8$; labour $= 0$.

Inequalities:	*acres*	*cost*	*labour*
	$12 \leq 60$	$2880 \leq 2880$	$1800 \leq 2400$

The Simplex Method: The Final Tableau

	Town x_1	*Single* x_2	*Two* x_3	*(Acres)* s_1	*(Cost)* s_2	*(Labour)* s_3	*Profit* p	*const*
acres	0	1/24	1/4	1	−1/240	0	0	48
cost	1	5/4	3/2	0	1/40	0	0	72
labor	0	−5/4	5/2	0	−5/8	1	0	600
obj fcn	0	3/4	5/2	0	3/8	0	1	1080

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- Shadow prices: acres $= 0$; cost $= 3/8$; labour $= 0$.

  Inequalities: *acres* $12 \leq 60$ *cost* $2880 \leq 2880$ *labour* $1800 \leq 2400$
- Reduced costs: Single story $= 3/4$; Two story $= 5/2$

The Simplex Method: The Final Tableau

	Town x_1	*Single* x_2	*Two* x_3	*(Acres)* s_1	*(Cost)* s_2	*(Labour)* s_3	*Profit* p	*const*
acres	0	1/24	1/4	1	−1/240	0	0	48
cost	1	5/4	3/2	0	1/40	0	0	72
labor	0	−5/4	5/2	0	−5/8	1	0	600
obj fcn	0	3/4	5/2	0	3/8	0	1	1080

- 72 Townhouses + 0 Single story + 0 Two story $= \$1{,}080{,}000$ profit.
- Shadow prices: acres $= 0$; cost $= 3/8$; labour $= 0$.

  Inequalities: *acres* $12 \leq 60$ *cost* $2880 \leq 2880$ *labour* $1800 \leq 2400$
- Reduced costs: Single story $= 3/4$; Two story $= 5/2$

Simplex Calculator

The Simplex Method: The "Big" LP

Time to ...

Apply the simplex method to the Big LP ▸ project.

Remember, there are:

12 decision variables $+\ 7$ constraints $+\ 12$ nonnegativity constraints!

So, we will have a linear system of

8 equations in 19 variables

The Simplex Method: The "Big" LP

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Apply the simplex method to the Big LP ▶ project.

Remember, there are:

12 decision variables $+\,7$ constraints $+\,12$ nonnegativity constraints!

So, we will have a linear system of

8 equations in 19 variables

Remember: there are nearly $50{,}400$ *potential vertices...*

The Simplex Method: The Big LP's Program

Minimize

$$w = 8x_{11} + 6x_{12} + 10x_{13} + 9x_{14} + 9x_{21} + 12x_{22} + 13x_{23} + 7x_{24} + 14x_{31} + 9x_{32} + 16x_{33} + 5x_{34}$$

subject to

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 35$$
$$x_{21} + x_{22} + x_{23} + x_{24} \leq 50$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 40$$
$$x_{11} + x_{21} + x_{31} \geq 45$$
$$x_{12} + x_{22} + x_{32} \geq 20$$
$$x_{13} + x_{23} + x_{33} \geq 30$$
$$x_{14} + x_{24} + x_{34} \geq 30$$
$$x_{ij} \geq 0$$

The Simplex Method: The Big LP's Final Tableau

The Final Tableau (#13) appears below.

x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	s_1	s_2	s_3	s_4	s_5	s_6	s_7	$-z$	
1	0	1	0	0	-1	0	-1	1	0	1	0	0	-1	0	-1	0	-1	0	0	25
-1	0	0	0	0	1	1	1	-1	0	0	0	0	1	0	1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	-1	0	0	0	0	45
0	1	0	1	0	1	0	1	-1	0	-1	0	1	1	0	1	0	1	0	0	10
0	0	0	-1	0	0	0	-1	1	1	1	0	-1	-1	0	-1	-1	-1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	-1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	3	0	0	9	9	13	5	1	-1020

The Simplex Method: The Big LP's Final Tableau

The Final Tableau (#13) appears below.

x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	s_1	s_2	s_3	s_4	s_5	s_6	s_7	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	−1	0	−1	0	−1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	1	0	1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	1	1	0	1	0	1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	−1	−1	0	−1	−1	−1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	3	0	0	9	9	13	5	1	−1020

Optimum: Cost $= 1020$ for $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$

The Simplex Method: The Big LP's Final Tableau

The Final Tableau (#13) appears below.

x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	s_1	s_2	s_3	s_4	s_5	s_6	s_7	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	−1	0	−1	0	−1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	1	0	1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	1	1	0	1	0	1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	−1	−1	0	−1	−1	−1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	3	0	0	9	9	13	5	1	−1020

Optimum: Cost $= 1020$ for $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$

Shadow prices: $s_1 = 3$, $s_4 = 9$, $s_5 = 9$, $s_6 = 13$, $s_7 = 5$ (s_2 and s_3 constraints are *nonbinding*)

The Simplex Method: The Big LP's Final Tableau

The Final Tableau (#13) appears below.

x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	s_1	s_2	s_3	s_4	s_5	s_6	s_7	$-z$	
1	0	1	0	0	−1	0	−1	1	0	1	0	0	−1	0	−1	0	−1	0	0	25
−1	0	0	0	0	1	1	1	−1	0	0	0	0	1	0	1	0	0	0	0	5
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	0	0	0	45
0	1	0	1	0	1	0	1	−1	0	−1	0	1	1	0	1	0	1	0	0	10
0	0	0	−1	0	0	0	−1	1	1	1	0	−1	−1	0	−1	−1	−1	0	0	10
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	−1	0	30
2	0	0	7	0	3	0	2	5	0	3	0	3	0	0	9	9	13	5	1	−1020

Optimum: Cost $= 1020$ for $x_{12} = 10$, $x_{13} = 25$, $x_{21} = 45$, $x_{23} = 5$, $x_{32} = 10$, $x_{34} = 30$

Shadow prices: $s_1 = 3$, $s_4 = 9$, $s_5 = 9$, $s_6 = 13$, $s_7 = 5$ (s_2 and s_3 constraints are *nonbinding*)

Reduced costs: $x_{11} = 2$, $x_{14} = 7$, $x_{22} = 3$, $x_{24} = 2$, $x_{31} = 5$, $x_{33} = 3$

The Simplex Method: The Dual LP

Definition (Primal & Dual Linear Programs)

A linear program has two forms:

Primal: maximize: $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

subject to:
$$\begin{cases} a_{11}\,x_1 + a_{12}\,x_2 + \cdots + a_{1n}\,x_n \le b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \cdots + a_{2n}\,x_n \le b_2 \\ \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots + a_{mn}\,x_n \le b_m \end{cases}$$

Dual: minimize: $w = b_1y_1 + b_2y_2 + \cdots + b_my_m$

subject to:
$$\begin{cases} a_{11}\,y_1 + a_{21}\,y_2 + \cdots + a_{m1}\,y_m \le c_1 \\ a_{12}\,y_1 + a_{22}\,y_2 + \cdots + a_{m2}\,y_m \le c_2 \\ \vdots \\ a_{1n}\,y_1 + a_{2n}\,y_2 + \cdots + a_{mn}\,y_m \le c_n \end{cases}$$

I.e., $\left\{\text{max: } z = \vec{c}\cdot\vec{x} \ \text{ s.t. } \mathbf{A}\vec{x} \le \vec{b}\right\} \iff \left\{\text{min: } w = \vec{b}\cdot\vec{y} \ \text{ s.t. } \mathbf{A}^T\vec{y} \ge \vec{c}\right\}$

▸ Example