Newton's Method Error Bound

Theorem (Newton[1]–Raphson Method[2] (1711))

Suppose f has 2 continuous derivatives on a neighborhood B of a root r. Set $x_{n+1} = x_n - f(x_n)/f'(x_n)$ and let $x_0 \in B(r,\delta)$. Then $x_n \to r$ and

$$|r - x_{n+1}| \leq c_\delta \, |r - x_n|^2 \, ;$$

that is, "x_n converges to r quadratically."

[1] In *De analysi per aequationes numero terminorum infinitas* (written 1669, pub. 1711). The 'derivative version' is due to Simpson (1740). See History.

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Further,

$$c_\delta = \frac{1}{2} \cdot \frac{\max\limits_{x \in B(r,\delta)} |f''(x)|}{\min\limits_{x \in B(r,\delta)} |f'(x)|}$$

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Proving the Error Bound

Proof (sketch).

1. Set $\varepsilon_n = r - x_n$.

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$$\varepsilon_{n+1} = r - \overbrace{\left[x_n - \frac{f(x_n)}{f'(x_n)}\right]}^{x_{n+1}} = \varepsilon_n + \frac{f(x_n)}{f'(x_n)}$$

$$= \frac{\varepsilon_n f'(x_n) + f(x_n)}{f'(x_n)} \tag{1}$$

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2. By Taylor's thm (with $a = x_n$ and $h = \varepsilon_n$):

$$0 = f(r) = f(x_n + \varepsilon_n) = f(x_n) + f'(x_n)\varepsilon_n + \tfrac{1}{2} f''(\xi_n)\,\varepsilon_n^2$$

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3. Put (2) into (1), then maximize the expression.[1] □

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Multiple Roots

Accelerating Convergence for Multiple Roots

If r is a root of multiplicity m, then set

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Try: $f(x) = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ with $r = 1$.