RK45 Pseudocode: Cheney & Kincaid

procedure *RK45*$(f, t, x, h, \varepsilon)$
real $\varepsilon, K_1, K_2, K_3, K_4, K_5, K_6, h, t, x, x_4$
external function f
real $c_{20} \leftarrow 0.25,\ c_{21} \leftarrow 0.25$
real $c_{30} \leftarrow 0.375,\ c_{31} \leftarrow 0.09375,\ c_{32} \leftarrow 0.28125$
real $c_{40} \leftarrow 12./13.,\ c_{41} \leftarrow 1932./2197.$
real $c_{42} \leftarrow -7200./2197.,\ c_{43} \leftarrow 7296./2197.$
real $c_{51} \leftarrow 439./216.,\ c_{52} \leftarrow -8.$
real $c_{53} \leftarrow 3680./513.,\ c_{54} \leftarrow -845./4104.$
real $c_{60} \leftarrow 0.5,\ c_{61} \leftarrow -8./27.,\ c_{62} \leftarrow 2.$
real $c_{63} \leftarrow -3544./2565.,\ c_{64} \leftarrow 1859./4104.$
real $c_{65} \leftarrow -0.275$
real $a_1 \leftarrow 25./216.,\ a_2 \leftarrow 0.,\ a_3 \leftarrow 1408./2565.$
real $a_4 \leftarrow 2197./4104.,\ a_5 \leftarrow -0.2$
real $b_1 \leftarrow 16./135.,\ b_2 \leftarrow 0.,\ b_3 \leftarrow 6656./12825.$
real $b_4 \leftarrow 28561./56430.,\ b_5 \leftarrow -0.18$
real $b_6 \leftarrow 2./55.$
$K_1 \leftarrow hf(t, x)$
$K_2 \leftarrow hf(t + c_{20}h, x + c_{21}K_1)$
$K_3 \leftarrow hf(t + c_{30}h, x + c_{31}K_1 + c_{32}K_2)$
$K_4 \leftarrow hf(t + c_{40}h, x + c_{41}K_1 + c_{42}K_2 + c_{43}K_3)$
$K_5 \leftarrow hf(t + h, x + c_{51}K_1 + c_{52}K_2 + c_{53}K_3 + c_{54}K_4)$
$K_6 \leftarrow hf(t + c_{60}h, x + c_{61}K_1 + c_{62}K_2 + c_{63}K_3 + c_{64}K_4 + c_{65}K_5)$
$x_4 \leftarrow x + a_1K_1 + a_3K_3 + a_4K_4 + a_5K_5$
$x \leftarrow x + b_1K_1 + b_3K_3 + b_4K_4 + b_5K_5 + b_6K_6$
$t \leftarrow t + h$
$\varepsilon \leftarrow |x - x_4|$
end procedure *RK45*

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procedure *RK45*$(f, t, x, h, \varepsilon)$
real $\varepsilon, K_1, K_2, K_3, K_4, K_5, K_6, h, t, x, x_4$
external function f
$K_1 \leftarrow hf(t, x)$
$K_2 \leftarrow hf(t + c_{20}h, x + c_{21}K_1)$
$K_3 \leftarrow hf(t + c_{30}h, x + c_{31}K_1 + c_{32}K_2)$
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$K_6 \leftarrow hf(t + c_{60}h, x + c_{61}K_1 + c_{62}K_2 + c_{63}K_3 + c_{64}K_4 + c_{65}K_5)$
$x_4 \leftarrow x + a_1K_1 + a_3K_3 + a_4K_4 + a_5K_5$
$x \leftarrow x + b_1K_1 + b_3K_3 + b_4K_4 + b_5K_5 + b_6K_6$
$t \leftarrow t + h$
$\varepsilon \leftarrow |x - x_4|$
end procedure *RK45*

RK45 Maple Code

Runge-Kutta Maple Function

$RK45\_Step := \mathbf{proc}(f, t, x, h)$
$\quad \mathbf{local}\ K, x4, x5, i, \varepsilon;$
$\quad \mathbf{global}\ c0, c, a, b;$
$\quad K_1 := h \cdot f(t, x);$
$\quad K_2 := h \cdot f\left(t + c0_2 \cdot h, x + c_{2,1} \cdot K_1\right);$
$\quad K_3 := h \cdot f\left(t + c0_3 \cdot h, x + c_{3,1} \cdot K_1 + c_{3,2} \cdot K_2\right);$
$\quad K_4 := h \cdot f\left(t + c0_4 \cdot h, x + c_{4,1} \cdot K_1 + c_{4,2} \cdot K_2 + c_{4,3} \cdot K_3\right);$
$\quad K_5 := h \cdot f\left(t + c0_5 \cdot h, x + c_{5,1} \cdot K_1 + c_{5,2} \cdot K_2 + c_{5,3} \cdot K_3 + c_{5,4} \cdot K_4\right);$
$\quad K_6 := h \cdot f\left(t + c0_6 \cdot h, x + c_{6,1} \cdot K_1 + c_{6,2} \cdot K_2 + c_{6,3} \cdot K_3 + c_{6,4} \cdot K_4 + c_{6,5} \cdot K_5\right);$
$\quad x4 := x + sum\left('a_i' \cdot 'K_i', i = 1..5\right);$
$\quad x5 := x + sum\left('b_i' \cdot 'K_i', i = 1..6\right);$
$\quad \varepsilon := \mathrm{abs}(x5 - x4);$
$\quad \mathbf{return}\left(t + h, x5, \varepsilon\right)$
$\quad \mathbf{end\ proc}:$

▸ RK45 Maple Worksheet

Adaptive RK45 Pseudocode

Adaptive Process Overview

Cheney & Kincaid (pg 453)

1. Given a step size h and an initial value $x(t)$, the *RK45* routine computes the value $x(t+h)$ and an error estimate ε.
2. If $\varepsilon_{\min} \leqq \varepsilon \leqq \varepsilon_{\max}$, then the step size h is not changed and the next step is taken by repeating step 1 with initial value $x(t+h)$.
3. If $\varepsilon < \varepsilon_{\min}$, then h is replaced by $2h$, provided that $|2h| \leqq h_{\max}$.
4. If $\varepsilon > \varepsilon_{\max}$, then h is replaced by $h/2$, provided that $|h/2| \geqq h_{\min}$.
5. If $h_{\min} \leqq |h| \leqq h_{\max}$, then the step is repeated by returning to step 1 with $x(t)$ and the new h value.

Burden & Faires (pg 233)

1. Given h and $x(t)$, use $RK45$ to compute $x(t+h)$ and ε.
2. Set $R = \left[|x_4 - x_5|/h\right]$ and $\delta = \sqrt[4]{\varepsilon_{max}/(2R)}$.
3. If $R \leq \varepsilon_{max}$, then output $(t+h, x5)$ and ε. Set $h = \delta h$ and go to Step 1. for the next point.
4. If $R > \varepsilon_{max}$, then set $h = \delta h$. Repeat Step 1. with the new h to recompute $x5$.

Adaptive RK45 Pseudocode

Adaptive Process: Cheney & Kincaid

Pseudocode

procedure *RK45_Adaptive*$(f, t, x, h, t_b, itmax, \varepsilon_{\max}, \varepsilon_{\min}, h_{\min}, h_{\max}, iflag)$
integer $iflag, itmax, n$; **external function** f
real $\varepsilon, \varepsilon_{\max}, \varepsilon_{\min}, d, h, h_{\min}, h_{\max}, t, t_b, x, x_{\text{save}}, t_{\text{save}}$
real $\delta \leftarrow \frac{1}{2} \times 10^{-5}$
output $0, h, t, x$
$iflag \leftarrow 1$
$k \leftarrow 0$
while $k \leqq itmax$
    $k \leftarrow k + 1$
    if $|h| < h_{\min}$ **then** $h \leftarrow \text{sign}(h)h_{\min}$
    if $|h| > h_{\max}$ **then** $h \leftarrow \text{sign}(h)h_{\max}$
    $d \leftarrow |t_b - t|$
    if $d \leqq |h|$ **then**
        $iflag \leftarrow 0$
        if $d \leqq \delta \cdot \max\{|t_b|, |t|\}$ **then** exit loop
        $h \leftarrow \text{sign}(h)d$
    end if
    $x_{\text{save}} \leftarrow x$
    $t_{\text{save}} \leftarrow t$
    call *RK45*$(f, t, x, h, \varepsilon)$
    output n, h, t, x, ε
    if $iflag = 0$ **then** exit loop
    if $\varepsilon < \varepsilon_{\min}$ **then** $h \leftarrow 2h$
    if $\varepsilon > \varepsilon_{\max}$ **then**
        $h \leftarrow h/2$
        $x \leftarrow x_{\text{save}}$
        $t \leftarrow t_{\text{save}}$
        $k \leftarrow k - 1$
    end if
end while
end procedure *RK45_Adaptive*