Secant Method Error Bound

Theorem (Secant Method[1])

Suppose f has 2 continuous derivatives on a neighborhood B of a root r. Set

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

and let $x_0 \in B(r, \delta)$. Then $x_n \to r$ and

$$\begin{aligned} |r - x_{n+1}| &\le c_\delta\, |r - x_n| \cdot |r - x_{n-1}| \\ |\varepsilon_{n+1}| &\le c_\delta |\varepsilon_n| \cdot |\varepsilon_{n-1}| \qquad (\le C \cdot |\varepsilon_n|^{\frac{1}{2}(1+\sqrt{5})}) \end{aligned}$$

that is, "x_n converges to r superlinearly."

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$$|r - x_{n+1}| \leq c_\delta \, |r - x_n| \cdot |r - x_{n-1}|$$
$$|\varepsilon_{n+1}| \leq c_\delta |\varepsilon_n| \cdot |\varepsilon_{n-1}| \qquad (\leq C \cdot |\varepsilon_n|^{\frac{1}{2}(1+\sqrt{5})})$$

that is, "x_n converges to r superlinearly."
Further,

$$c_\delta = \frac{1}{2} \cdot \frac{\max_{x \in B(r,\delta)} |f''(x)|}{\min_{x \in B(r,\delta)} |f'(x)|}$$

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Proving the Error Bound, I

Proof (sketch).

1. Set $\varepsilon_n = r - x_n$.

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$$\varepsilon_{n+1} = r - \overbrace{\left[\frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}\right]}^{x_{n+1}}$$

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$$= \frac{(r - x_{n-1})f(x_n) - (r - x_n)f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

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So

$$\varepsilon_{n+1} = \frac{1}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} \cdot \frac{\frac{f(x_n)}{\varepsilon_n} - \frac{f(x_{n-1})}{\varepsilon_{n-1}}}{x_n - x_{n-1}} \cdot \varepsilon_{n-1}\varepsilon_n$$

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Whence

$$\varepsilon_{n+1} \approx \frac{1}{f'(r)} \cdot \frac{\frac{f(x_n)}{\varepsilon_n} - \frac{f(x_{n-1})}{\varepsilon_{n-1}}}{x_n - x_{n-1}} \cdot \varepsilon_{n-1}\varepsilon_n \qquad (1)$$

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Proving the Error Bound, II

(Proof (continued).)

2. By Taylor's thm (with $a = r$ and $h = -\varepsilon_n$):

$$f(x_n) = f(r - \varepsilon_n) = \underbrace{f(r)}_{=0} - f'(r)\varepsilon_n + \tfrac{1}{2} f''(r)\,\varepsilon_n^2 + O(\varepsilon_n^3)$$

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Subtract (3) from (2), then divide by $x_n - x_{n-1}$ to obtain

$$\frac{\frac{f(x_n)}{\varepsilon_n} - \frac{f(x_{n-1})}{\varepsilon_{n-1}}}{x_n - x_{n-1}} \approx \tfrac{1}{2} f''(r)\,\frac{\varepsilon_n - \varepsilon_{n-1}}{x_n - x_{n-1}}$$

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3. Put (4) into (1), then maximize the expression.