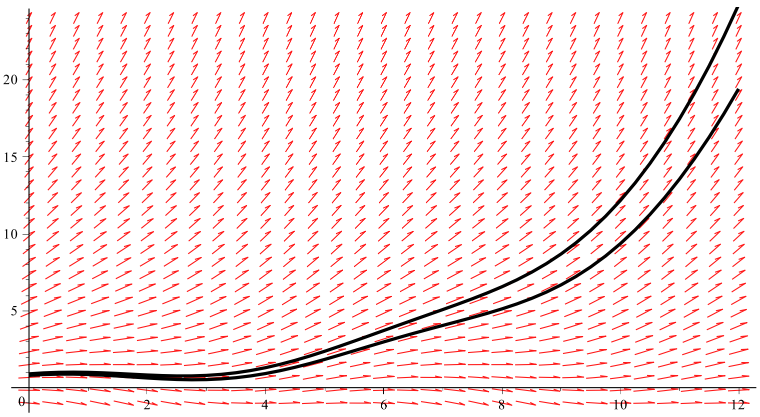
First order IVP

$$\begin{cases} x' = f(t,x) \\ x(t_0) = s \end{cases} \tag{1}$$

Divergence Example

$$\begin{cases} x' = \frac{1}{3}x - \frac{1}{2}\sin(t) \\ x(0) = s \end{cases} \tag{2}$$

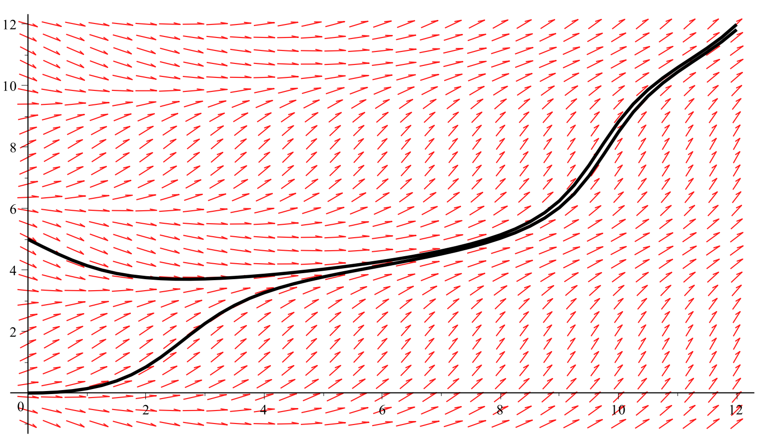
Let $s = 0.8$ and 0.9.



Convergence Example

$$\begin{cases} x' = \sin(x) + \frac{1}{5}t \\ x(0) = s \end{cases} \tag{3}$$

Let $s = 0$ and 5.



How to Differentiate the Cases

Suppose there is no truncation error, only round-off in the initial value. (E.g., $x_0 = 1/3$ cannot be exact in floating point.) Then

$$\textit{total error} = \textit{local error} + \textit{accumulated error}$$

Analysis w.r.t. the Initial Value

Consider s as a variable in $x_0 = x$. Then

$$\frac{\partial x}{\partial s} \approx \frac{x(t,s+h)-x(t,s)}{h}$$

So

$$x(t,s+h)-x(t,s) \approx h \cdot \frac{\partial x}{\partial s}$$

Whence the solutions diverge when $\frac{\partial x}{\partial s} \to \infty$ as $t \to \infty$ (for fixed h). And the solutions converge when $\frac{\partial x}{\partial s} \to const$ as $t \to \infty$ (for fixed h).

Calculating $\partial x/\partial s$

Let $x(t) = x(t,s)$. Then

$$\frac{\partial x}{\partial t} = f(t,x(t,s))$$

So

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial x}{\partial t} &= \frac{\partial}{\partial s} f(t,x(t,s)) \\ &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial s}\end{aligned}$$

Since t & s are independent of each other, $\partial t/\partial s = 0$. Thence

$$\frac{\partial}{\partial s}\frac{\partial x}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$$

Interchange order on the left to see

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}$$

Set $u = x_s$ and $q = f_x$. Thus

$$\frac{\partial}{\partial t} u = q u$$

This is a separable DE with solution $u(t) = c_0 \exp\left(\int_{t_0}^{t} q(\tau)d\tau\right)$.

Now $u \underset{t}{\to} \infty$ when $q \geq \delta$ since then $\int_{t_0}^{t} q(\tau)d\tau \geq \delta(t-t_0)$. Whereupon the condition we're looking for is

$$\text{If } f_x(t,x) \geq \delta > 0, \text{ the solutions diverge; i.e., the solutions are unstable} \qquad (*)$$