Taylor Series: Two Variables

Theorem (Taylor Series in Two Variables)

Suppose f is sufficiently continuously differentiable in a neighborhood of (x_0, y_0). Then

$$f(x_0+h, y_0+k) = \sum_{j=0}^{N} \frac{1}{j!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^j f(x_0,y_0) + O\Big((h,k)^{N+1}\Big)$$

Examples

Let $\vec{u}_0 = \langle x_0, y_0 \rangle$.

- First order: $f(\vec{u}_0 + \langle h,k\rangle) = f(\vec{u}_0) + \left[f_x(\vec{u}_0)h + f_y(\vec{u}_0)k\right]$
- Second order: $f(\vec{u}_0 + \langle h,k\rangle) = f(\vec{u}_0) + \left[f_x(\vec{u}_0)h + f_y(\vec{u}_0)k\right]$

$$+\tfrac{1}{2}\left[f_{xx}(\vec{u}_0)h^2 + 2f_{xy}(\vec{u}_0)hk + f_{yy}(\vec{u}_0)k^2\right]$$

- Third order: $f(\vec{u}_0 + \langle h,k\rangle) = f(\vec{u}_0) + \left[f_x(\vec{u}_0)h + f_y(\vec{u}_0)k\right]$

$$+\tfrac{1}{2}\left[f_{xx}(\vec{u}_0)h^2 + 2f_{xy}(\vec{u}_0)hk + f_{yy}(\vec{u}_0)k^2\right]$$

$$+\tfrac{1}{3!}\left[f_{xxx}(\vec{u}_0)h^3 + 3f_{xxy}(\vec{u}_0)h^2k + 3f_{xyy}(\vec{u}_0)hk^2 + f_{yy}(\vec{u}_0)k^3\right]$$