Taylor's Theorem

Theorem (Taylor's Theorem[1] (1715 {*first in a 1712 letter*}))

Suppose f *has* $(n+1)$ *continuous derivatives on a neighborhood of* c*. Then* $f(x) = T_n(x) + E_{n+1}$ *where*

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}\,(x-c)^k$$

and[2]

$$E_{n+1} = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\,(x-c)^{n+1}$$

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for some ξ_x *(depending on* x*) between* c *and* x*; or*

$$|E_{n+1}| \le \frac{1}{(n+1)!} \cdot M_{n+1} \cdot |x-c|^{n+1}$$

where $M_{n+1} \ge \max_{t\in N(c,x)} \left| f^{(n+1)}(t) \right|$.

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Proving Taylor's Theorem

Proof[1] (sketch).

1. The *FToC* $\implies f(x) = f(c) + \int_0^{x-c} f'(x-t)\,dt$.

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3. Repeatedly integrate by parts with $u = f^{(k)}(x-t)$; $dv = \frac{1}{(k-1)!}t^{k-1}\,dt$ to find:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + E_{n+1}$$

where

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4. Apply the First Mean Value Theorem for Integrals to finish. □

[1] See Taylor's proof.

Taylor's Theorem, Alternate Version

Theorem (Taylor's Theorem for $(c+h)$)

Suppose f *has* $(n+1)$ *continuous derivatives on a neighborhood of* c*.*
Then $f(c+h) = T_n(c+h) + E_{n+1}$ *where*

$$T_n(c+h) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} h^k$$

and

$$E_{n+1} = \frac{f^{(n+1)}(\xi_h)}{(n+1)!} h^{n+1} = O\big(h^{n+1}\big)$$

$$|E_{n+1}| \leq \frac{M_{n+1}}{(n+1)!} \cdot |h|^{n+1}$$

for some ξ_h *(depends on* h*) between* c *&* $c+h$ *and* $M_{n+1} \geq \max_{t \in N} \left| f^{(n+1)}(t) \right|$*.*

Alternate Forms of the Remainder

Forms of the Remainder[2]

Lagrange (1797):
$$R_n(x) = \frac{1}{(n+1)!}\, f^{(n+1)}(c_x)\,(x-a)^{n+1}$$
for some c_x between x and a.

Cauchy (1821):
$$R_n(x) = \frac{1}{n!}\, f^{(n+1)}(c_x)\,(x-a)(x-c)^n$$
for some c_x between x and a.

Integral Form:
$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-t)^n\,dt$$

Uniform Estimate:
$$R_n(x) = \max_{x\in B}\left|f^{(n+1)}(x)\right| \cdot \frac{r^{n+1}}{(n+1)!}$$
for all x in $B = B(a,r)$

[2]See, e.g., Whitaker & Watson, *A Course of Modern Analysis*, Cambridge, 1927. Also see *Schlömilch remainder*.