Intro to Linear Algebra
MAT 5230, §101

Wm C Bauldry

Autumn Semester, 2005

Algebraic Structures

Definition 1 *A* Group *is a pair* $\{X; \cdot\}$ *such that*

1. "$\cdot$" *is* closed *on* X.
2. "$\cdot$" *is* associative *on* X.
3. *There is an identity* $e \in X$ *(w.r.t. "$\cdot$").*
4. *Every element* $a \in X$ *has an inverse* a^{-1} *(w.r.t. "$\cdot$").*

Definition 2 *A* Ring *is a triple* $\{X; +, \cdot\}$ *such that*

1. $\{X; +\}$ *is an* Abelian group.
2. $\{X; \cdot\}$ *is a* semigroup *(lacks identity and inverses).*
3. "$\cdot$" *distributes over* "$+$".

Algebraic Structures

Definition 3 *A* Field *is a triple* $\{X; +, \cdot\}$ *such that*

1. $\{X; +, \cdot\}$ *is a* ring.
2. $\{X^{\#}; \cdot\}$ *is an* Abelian group *where* $X^{\#} = X - \{0\}$.

Definition 4 *A* Vector Space *is an Abelian group* $\{X; +\}$ *over a field* $\{F; +, \cdot\}$ *with a* scalar product $F \times X \to X$. *For* $\alpha, \beta \in F$ *and* $x, y \in X$*,*

1. $\alpha(x + y) = \alpha x + \alpha y$
2. $(\alpha + \beta)x = \alpha x + \beta x$
3. $(\alpha\beta)x = \alpha(\beta x)$
4. $1x = x$

Field

Definition 3 (Field) *Let $F \neq \emptyset$ be a set with addition "+":$X \times X \to X$ and multiplication "$\cdot$":$F \times X \to X$. Then $\{F; +, \cdot\}$ with the operations forms a* field *if the following axioms are satisfied:*

1. $x + y = y + x,\ x \cdot y = y \cdot x$ — commutative laws
2. $x + (y + z) = (x + y) + z,\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$ — associative laws
3. *There is a unique element* 0 *satisfying* $0 + x = x$ — additive identity
4. *To each* x, $\exists$ *a unique* $-x$ *such that* $x + (-x) = 0$ — additive inverse
5. *There is a unique element* 1 *satisfying* $1 \cdot x = x$ — mult. identity
6. *To each* $x \neq 0$, $\exists$ *a unique* x^{-1} *such that* $x \cdot x^{-1} = 1$ — mult. inverse
7. $x \cdot (y + z) = x \cdot y + x \cdot z$ — "$\cdot$" over "+" distributive law

Examples of Fields

1. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.
2. $\mathbb{Z}$ is *not* a field. (*Why?*)
3. Let p be a prime. Then $\mathbb{Z}_p$ is a p-element field.
4. $\mathbb{Q}\left[\sqrt{2}\right] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field.
5. $\mathbb{Z}\left[\sqrt{2}\right] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is *not* a field. (*Why?*)
6. $\mathbb{Q}\left[\sqrt[3]{3}\right] = \{a + b\sqrt[3]{3} + c\sqrt[3]{3^2} \mid a, b, c \in \mathbb{Q}\}$ is a field.
7. $\mathbb{Z}_p[i]$, p is prime, is a field (with p^2 elements).

Vector Space

Definition 4 (Vector Space) *Let $X \neq \emptyset$ be a set* (vectors) *and F be a field* (scalars) *with* vector addition "+":$X \times X \rightarrow X$ *and* scalar multiplication "·":$F \times X \rightarrow X$. *Then X and F with the operations forms a* vector space *(or* linear space*), "X is a* vector space *over F," if the following axioms are satisfied:*

1. $x + y = y + x$ — commutative law
2. $x + (y + z) = (x + y) + z$ — associative law
3. *There is a unique vector* 0 *satisfying* $0 + x = x$ — 'zero vector,' identity
4. $\alpha(x + y) = \alpha x + \alpha y$ — scalar "·" over vector "+" distributive law
5. $(\alpha + \beta)x = \alpha x + \beta x$ — scalar "+" over scalar "·" distributive law
6. $(\alpha\beta)x = \alpha(\beta x)$ — scalar homogeneity
7. $0x = 0$ — scalar-vector additive identity relation (*implied by 5.*)
8. $1x = x$ — scalar-vector multiplicative identity relation

Examples of Vector Spaces

1. Let $n \in \mathbb{Z}^+$. Then $\mathbb{Q}^n$, $\mathbb{R}^n$, and $\mathbb{C}^n$ are vector spaces.
2. Let $n \in \mathbb{Z}^+$. Then $\mathbb{P}^n$, the polynomials (real or complex) of degree less than or equal to n, forms a vector space.
3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is a vector space.
4. Let F be a field and $n \in \mathbb{Z}^+$. Then F^n is a vector space.
5. Let $M_{m\times n}$ be the $m \times n$ matrices with entries in a field F with componentwise addition and scalar multiplication.
6. Let $K \subseteq \mathbb{R}$ be a closed interval. Then $C(K)$, the continuous real-valued functions on K form a vector space.
7. Let $O \subseteq \mathbb{R}$ be an open interval. Then $C^1(O)$, the continuously differentiable real-valued functions on O form a vector space.

Homomorphisms

Definition 5 (Group Homomorphism) *Let* $\{X;+_X\}$ *and* $\{Y;+_Y\}$ *be two groups with* $\rho : X \to Y$. *Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

Definition 6 (Ring Homomorphism) *Let* $\{X;+_X,\cdot_X\}$ *and* $\{Y;+_Y,\cdot_Y\}$ *be two rings with* $\rho : X \to Y$. *Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

$$\rho(x_1 \cdot_X x_2) = \rho(x_1) \cdot_Y \rho(x_2)$$

Vector Space Homomorphism

Definition 7 (Linear Transformation) *Let X and Y be vector spaces over the same field F. Then the relation $\rho : X \to Y$ is a* linear transformation *if and only if for every $\alpha \in F$ and $x_1, x_2 \in X$, it follows that:*

(1) $$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

(2) $$\rho(\alpha \cdot x_1) = \alpha \cdot \rho(x_1)$$

Linear Transformation

(1)

$$\begin{array}{ccc} [x_1, x_2] & \xrightarrow{+} & x_1 + x_2 \\ \rho\big\downarrow & & \rho\big\downarrow \\ [\rho(x_1), \rho(x_2)] & \xrightarrow{+} & \begin{array}{c}\rho(x_1 + x_2) = \\ \rho(x_1) + \rho(x_2)\end{array} \end{array}$$

(2)

$$\begin{array}{ccc} [\alpha, x_1] & \xrightarrow{\cdot} & \alpha \cdot x_1 \\ \rho\big\downarrow & & \rho\big\downarrow \\ [\alpha, \rho(x_1)] & \xrightarrow{\cdot} & \begin{array}{c}\rho(\alpha \cdot x_1) = \\ \alpha \cdot \rho(x_1)\end{array} \end{array}$$