Homomorphisms

Definition 1 (Group Homomorphism) *Let* $\{X; +_X\}$ *and* $\{Y; +_Y\}$ *be two groups with* $\rho : X \to Y$. *Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

Definition 2 (Ring Homomorphism) *Let* $\{X; +_X, \cdot_X\}$ *and* $\{Y; +_Y, \cdot_Y\}$ *be two rings with* $\rho : X \to Y$. *Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

$$\rho(x_1 \cdot_X x_2) = \rho(x_1) \cdot_Y \rho(x_2)$$

Vector Space Homomorphism

Definition 3 (Linear Transformation) *Let X and Y be vector spaces over the same field F. Then the relation $\rho : X \to Y$ is a* linear transformation *if and only if for every $\alpha \in F$ and $x_1, x_2 \in X$, it follows that:*

(1) $$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

(2) $$\rho(\alpha \cdot x_1) = \alpha \cdot \rho(x_1)$$

Examples

1. Set $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ by $\phi(x, y) = (x, 0, 0, y)$.
2. Set $\psi : \mathbb{R}^2 \to \mathbb{C}$ by $\psi(x, y) = x + i\, y$.

Linear Transformation

(1)

$$\begin{array}{ccc} [x_1, x_2] & \xrightarrow{+} & x_1 + x_2 \\ \rho \downarrow & & \rho \downarrow \\ [\rho(x_1), \rho(x_2)] & \xrightarrow{+} & \rho(x_1 + x_2) = \\ & & \rho(x_1) + \rho(x_2) \end{array}$$

(2)

$$\begin{array}{ccc} [\alpha, x_1] & \xrightarrow{\cdot} & \alpha \cdot x_1 \\ \rho \downarrow & & \rho \downarrow \\ [\alpha, \rho(x_1)] & \xrightarrow{\cdot} & \rho(\alpha \cdot x_1) = \\ & & \alpha \cdot \rho(x_1) \end{array}$$