Subspace of a Vector Space

Definition 1 (Subspace) *Let X be a vector space over F and let $\emptyset \neq V \subseteq X$. Then V is a* subspace *of X iff*

1. *$\forall u, v \in V$, we have $u + v \in V$* *(closed under addition)*
2. *$\forall \alpha \in F$, $\forall u \in V$, we have $\alpha u \in V$* *(closed under scalar mult.)*

Theorem 1 *A subspace of a vector space is itself a vector space.*

Proof. Let V be a subspace of X. V is closed under vector addition and scalar multiplication by definition. All remaining vector space properties — with the exception of $0 \in V$ — are inherited from X. Let $v \in V$ (because $V \neq \emptyset$). Since $0 \in F$, then $0v = 0 \in V$. Thus V is a vector space. □

Note. Every vector space has at least 2 subspaces. What are they?

Examples of Subspaces

- $\{0\}$ and X are always subspaces of X
- $\mathbb{R}^2$ is a subspace[a] of $\mathbb{R}^3$, $\mathbb{C}^2$ is a subspace of $\mathbb{C}^3$.
- For $m < n$, we have that $\mathbb{R}^m$ is a subspace of $\mathbb{R}^n$
- For $m < n$, we have that $\mathbb{P}^m$ is a subspace of $\mathbb{P}^n$
- Is
  - $V_1 = \{(x, 1) \,|\, x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?
  - $V_2 = \{(x, y, x + y, 0) \,|\, x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^4$?
  - $V_3 = \{(x, y, x + y + 2, 0) \,|\, x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^4$?

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[a] Thinking of $\mathbb{R}^2$ as a subset such as $\{(x, y, 0) \,|\, x, y \in \mathbb{R}\}$, &c., of $\mathbb{R}^3$. Formally, $\mathbb{R}^2$ is isomorphic to a subspace of $\mathbb{R}^3$.