Operations with Subspaces

Theorem 1 *Let* X *be a vector space over* F *and let* V_1 *and* V_2 *be subspaces of* X*. Then* $V = V_1 \cap V_2$ *is a subspace.*

Pf. (Exercise.)

Theorem 2 *Let* X *be a vector space over* F *and let* X_i *for* $i \in I$ *be subspaces of* X *where* I *is some index set. Then* $V = \bigcap_{i \in I} X_i$ *is a subspace.*

Pf. (Easy closure calculations.)

NB: Unions (usually) or complements of subspaces do not form new subspaces.

Direct Sum

Definition 1 (Direct Sum) *Let* $X_1, X_2, \ldots, X_r$ *be subspaces of* X. *The set* $X_1 + X_2 + \cdots + X_r$ *forms the* direct sum $X_1 \oplus X_2 \oplus \cdots \oplus X_r$ *iff for every* x *in the sum, there is a unique set of* $x_i \in X_i$ *such that* $x = \sum_{i=1}^{r} x_i$.

Theorem 3 $X_1 + X_2 = X_1 \oplus X_2$ *if and only if* $X_1 \cap X_2 = \{0\}$.

Pf. Based on: Let $0 \neq v \in X_1 \cap X_2$. Then $v = v + 0 = 0 + v$ is two different ways to write v.

Note. $X_1 + X_2$ is a subspace; $X_1 \oplus X_2$ is a subspace that 'looks like' a direct product.

Subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

Example 1 *Set $X = \mathbb{R}^2$. Let X_1 be given by the line $y = x$ and X_2 by the line $y = -x$. Then*

$$\{0\} = X_1 \cap X_2 \quad \subseteq \quad X_1 \cup X_2 \quad \subseteq \quad X_1 + X_2 \quad = \quad X_1 \oplus X_2 = \mathbb{R}^2$$

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Example 2 *Set $X = \mathbb{R}^3$. The subspaces of $\mathbb{R}^3$ are:*

- $\{0\}$
- *A line L through the origin.*
- *The direct sum of two distinct lines through the origin $L_1 \oplus L_2$ yields a plane.*
- *The direct sum of three distinct non-coplanar lines through the origin $L_1 \oplus L_2 \oplus L_3$ yields $\mathbb{R}^3$.*