Linear Combinations

Note: From now on, α_i, etc., will be elements of the base field F and x_i, y_i, etc., will be vectors from the space X.

Definition 1 (Finite Linear Combination) *Let $Y \subseteq X$. A vector $x \in X$ is a* (finite) linear combination *of vectors in Y iff there is a finite set of vectors $\{y_i\} \subseteq Y$ and scalars $\{\alpha_i\}$ such that*

$$x = \sum_{i=1}^{n} \alpha_i \, y_i$$

Note: The sum is not required to be unique. (Unlike $\oplus$.)

Example 1 *Let $Y = \{(1,0),(1,1),(0,1)\} \subset \mathbb{R}^2$. Then the vector $x = (2,3)$ can be written as $x = 2(1,0) + 3(0,1)$ or as $x = 2(1,1) + 1(0,1)$ or as $x = -1(1,0) + 3(1,1)$.*

Generated Subspace & Span

Theorem 1 *Let* $\emptyset \neq Y \subseteq X$*. Define*

$$V(Y) \triangleq \{\textit{all linear combinations from } Y\}.$$

Then $V(Y)$ *is a subspace of* X *and is called the* subspace generated by Y*.*

Definition 2 (Span) Y spans X *if and only if* $V(Y) = X$*.*

Example 2 *Let* $Y = \{(1,0),(1,1),(0,1)\}$*. Then* Y *spans* $\mathbb{R}^2$*. (Exercise.)*

Example 3 *Let* $Z = \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$*. Does the set* Z *span* $\mathbb{R}^3$*?*

Dependence and Independence

Definition 3 (Linear Dependence) *Let $\{x_1, x_2, \ldots, x_m\}$ be a nonempty subset of X. If there exists a set of scalars $\{\alpha_i\}$, not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m = 0$, then $\{x_1, x_2, \ldots, x_m\}$ is* linearly dependent.

Definition 4 (Linear Independence) *If the nonempty subset $\{x_1, x_2, \ldots, x_m\}$ of X is not linearly dependent, then $\{x_1, x_2, \ldots, x_m\}$ is* linearly independent.

Example 4 *Y and Z from the previous examples are both linearly dependent.*

Example 5 *Let $W = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. Then W is linearly independent.*

More Examples

Example 6 *Let* $V = \{(1,1,0,0),(1,0,1,0),(1,1,1,0)\}$*. Is* V *linearly independent? Does* V *span* $\mathbb{R}^4$*?*

Example 7 *Let* $U = \{(0,0,0),(1,0,0),(0,1,0)\}$*. Is* U *linearly independent?*

Example 8 *Let* $\mathcal{P} = \{1, x, x^2, x^3, \ldots\}$*. Then* $V(\mathcal{P}) = \mathbb{P}$*, the set of all real polynomials; i.e.,* $\mathcal{P}$ *spans* $\mathbb{P}$*. Is* $\mathcal{P}$ *linearly independent? Yes! But how do we show this? Consider*

$$p(x) = \sum_{i=0}^{n} \alpha_i x_i = 0$$

and note that the only n*th degree polynomial with* $n+1$ *roots, is* $p(x) \equiv 0$*.*