# Linear Independence

**Theorem 1 (Uniqueness)** *Let* $Y = \{x_1, x_2, \ldots, x_m\}$ *be a linearly independent set of vectors. If* $\sum_{i=1}^m \alpha_i x_i = \sum_{i=1}^m \beta_i x_i$, *then* $\alpha_i = \beta_i$ *for* $i = 1..m$.

**Pf**. Simple calculation.

**Theorem 2** *A set* $Y$ *is linearly dependent if and only if some vector* $x \in Y$ *can be written as a linear combination of other vectors in* $Y$.

- Add any number of vectors to a dependent set, it will still be dependent.
- Add one vector to an independent set, it may or may not stay independent.

# Infinite Example

**Example 1** *Let* $\mathcal{P} = \{1, x, x^2, x^3, \ldots\}$*. Then* $V(\mathcal{P}) = \mathbb{P}$*, the set of all real polynomials; i.e.,* $\mathcal{P}$ *spans* $\mathbb{P}$*. Is* $\mathcal{P}$ *linearly independent? Yes! But how do we show this? Let* $p(x) \in \mathbb{P}$*. Then, for some* $n$*,*

$$p(x) = \sum_{i=0}^{n} \alpha_i x_i = 0.$$

*Note that the only* $n$*th degree polynomial with* $n+1$ *roots, is* $p(x) \equiv 0$*. Hence all* $\alpha_i$ *are* $0$*.*

# Unique Expression

**Theorem 3 (Uniqueness of Expression)** *A finite nonempty set $Y$ is linearly independent if and only if, for each nonzero $y \in V(Y)$, there exists a unique subset $\{x_1, \ldots, x_m\}$ of $Y$ and a unique set of scalars $\{\alpha_1, \ldots, \alpha_m\}$ such that $y = \sum_{i=1}^{m} \alpha_i x_i$.*

**Assignment**:

1. Prove Theorem 2
2. Prove Theorem 3

**Theorem 4** *$Y$ is linearly independent if and only if $Z \subsetneq Y$ implies $V(Z) \neq V(Y)$.*

**Pf**. Exercise.

# Basis of a Vector Space

**Definition 1 (Hamel Basis)** *A (finite) set* $Y \subseteq X$ *is a* Hamel basis *(or just a* basis*) if and only if*

1. $Y$ *is linearly independent*
2. $V(Y) = X$

Id est*,* $Y$ *is a (finite) linearly independent spanning set.*

**Theorem 5** *If* $Y$ *is linearly independent, then* $Y$ *is a basis for* $V(Y)$*.*

**Pf**. Exercise.

**Note**: The theorem *Every vector space has a basis* is a result of the *Axiom of Choice*.