Basis of a Vector Space

Recall:

Definition 1 (Hamel Basis) *A set* $Y \subseteq X$ *is a* Hamel basis *(or just a* basis*) if and only if*

1. Y *is linearly independent*
2. $V(Y) = X$

Note: The theorem «*Every vector space has a basis*» is a result of the *Axiom of Choice*.

Exempli gratia

- $\{(0,1),(1,2)\}$ is a basis of $\mathbb{R}^2$
- $\{(1,1,0),(1,0,1),(0,1,1)\}$ is a basis of $\mathbb{R}^3$
- $\{(1,1,0),(1,2,0),(2,1,0)\}$ is *not* a basis of $\mathbb{R}^3$

Basis Properties

Theorem 1 (Uniqueness of Scalars) *Let* $\{x_1, x_2, \ldots, x_n\}$ *be a basis for* X*. Then for each vector* $x \in X$*, there is a unique set of scalars* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *such that*

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

Pf. Standard calculation.

Theorem 2 (Maximum Independent Set Size) *Suppose that* $B = \{x_1, x_2, \ldots, x_n\}$ *is a basis of* X *with* n *finite and* $Y = \{y_1, y_2, \ldots, y_m\}$ *is a set of linearly independent vectors. Then* $m \leq n$*.*

Note: n is *finite* is necessary.

Proof of Theorem 2 - Outline

Proof Outline.

1. Assume $m > n$.
2. Write y_1 as a linear combination of the x_i. At least one coefficient can't be 0, say the coefficient of x_n (reindex x's if necessary).
3. Replace x_n in B with y_1. Show B still is a basis for X.
4. Start over with y_2 and the "new" B. Replace x_{n-1} by y_2.
5. Continue the process until y_n replaces x_1.
6. B - still a basis - now is $\{y_1, y_2, \ldots, y_n\}$.
7. Thus y_{n+1} can be written as as linear combination from B contradicting the linear independence of Y. Hence $m \leq n$.

Dimension

Theorem 3 *If* $B = \{x_1, x_2, \ldots, x_N\}$ *is a basis of* X *for some* $N < \infty$*, then every basis of* X *contains exactly* N *vectors.*

Pf. • Let B_1 be a basis with n vectors and B_2 be a basis with m vectors.
• Apply Theorem 2 with B_1 as the basis and B_2 as the linearly independent set. Therefore $m \leq n$.
• Now apply Theorem 2 with B_2 as the basis and B_1 as the linearly independent set. Therefore $n \leq m$.
• Since $m \leq n$ and $n \leq m$, it follows that $m = n$.