Dimension of a Vector Space

Definition 1 (Dimension) *If* X *has a finite basis of* n *vectors, then* X *is* finite dimensional *and has* dimension $dim(X) = n$. *If* X *is not finite dimensional, then* X *has* infinite dimension *and* $dim(X) = \infty$.

Example 1 *Several standard spaces:*

- $dim(\mathbb{R}^n) = n$
- $dim(\mathbb{P}^n) = n + 1$
- $dim(\mathbb{R}^\infty) = \infty$ *The space of real sequences is large (but it's a "small* ∞*")*
- $dim(\mathbb{P}) = \infty$ *(another "small* ∞*," isomorphic to* $\mathbb{R}^\infty$*)*

Examples

Example 2 *Infinite dimensional spaces*

- $dim(\mathcal{C}[0,1]) = \infty$ *The space of continuous functions on* $[0,1]$ *is very large (a "big* ∞*")*
- $dim(\mathcal{B}(\mathbb{R})) = \infty$ *with* $\mathcal{B}(\mathbb{R}) = \{$*bounded real functions* $\}$
- *Is the following true:*
  Let Z *be an arbitrary set and* X *an arbitrary vector space over* F*. The space of all functions from* Z *to* X*, written* X^Z*, is a vector space over* F *with dimension* $dim(X^Z) = dim(X)^{|Z|}$

Basis & Dimension Facts

Basis Facts

- Every vector space has a basis (requires the Axiom of Choice)
- Every linearly independent set can be extended to a basis
- A linearly independent set can be no larger than a basis
- A set containing more vectors than a basis must be linearly dependent
- Any two bases for a vector space contain the same number of vectors (finite dimensional case)
- If X has a set with n linearly independent vectors and every set of $n+1$ vectors is dependent, then $dim(X) = n$
- If Y is a subspace of X, then $dim(Y) \leq dim(X)$.

“Two Out of Three Ain’t Bad”

Theorem 1 *Suppose* X *is a vector space with* $dim(X) = n$ *and* $Y \subseteq X$. *If any two of the following hold, then the third also holds.*

1. Y *spans* X
2. Y *is linearly independent*
3. Y *contains exactly* n *vectors*

Theorem 2 *Suppose that* $dim(X) < \infty$ *and that* $X = Y \oplus Z$. *Then* $dim(X) = dim(Y) + dim(Z)$.

Nota Bene: Recall that $\oplus$ is the “interior analogue” of $\times$ and that if $X = Y \times Z$, then $dim(X) = dim(Y) \times dim(Z)$.

“Sum of Dimensions” Proof

Proof of Theorem 2 (3.3.43).

Since $dim(X) < \infty$, so are $dim(Y)$ and $dim(Z)$. Therefore there are bases of Y and Z: $\mathcal{B}_Y = \{y_1, \ldots, y_n\}$ and $\mathcal{B}_Z = \{z_1, \ldots, z_m\}$. Set $\mathcal{B} = \mathcal{B}_Y \cup \mathcal{B}_Z$. Let

$$0 = \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{m} \beta_i z_i$$

be a linear combination from $\mathcal{B}$. Since representation of vectors is unique in $X = Y \oplus Z$, we have that $0 = \sum_{i=1}^{n} \alpha_i y_i$ and $0 = \sum_{i=1}^{m} \beta_i z_i$ Therefore $0 = \alpha_i = \beta_j$ for all i and j as $\mathcal{B}_Y$ and $\mathcal{B}_Z$ are independent. I.e., $\mathcal{B}$ is linearly independent.

Since $X = Y \oplus Z$, it is clear that $\mathcal{B}$ spans X. Hence, $|\mathcal{B}| = n + m = dim(X)$.