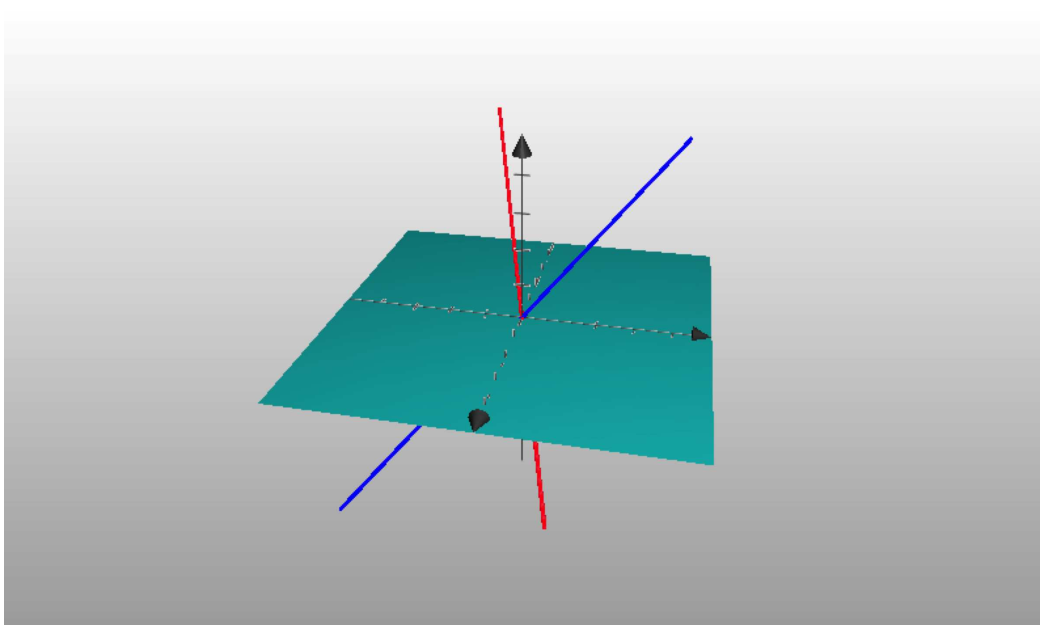
Subspaces and Direct Sums

The theorem *Every linearly independent set can be extended to a basis* has:

Corollary 1 *Suppose X is an n-dimensional vector space with an m-dimensional subspace Y. Then there exists a subspace Z of dimension $(n-m)$ such that $X = Y + Z$.*

Pf. (Sketch) Take bases $\mathcal{B}_X$ for X and $\mathcal{B}_Y$ for Y. Eliminate the portion of $\mathcal{B}_X$ dependent on $\mathcal{B}_Y$. The remaining vectors form a basis for Z.

Note: Z need not be unique. (Z= red or blue)



Linear Transformations

Definition 1 (Linear Transformation) *A mapping T from a vector space X into a vector space Y, both spaces over the field F, is a* linear transformation*, written as $T \in L(X, Y)$, if and only if for all $x \in X$, $y \in Y$, and $\alpha \in F$, we have*

1. $T(x + y) = T(x) + T(y)$
2. $T(\alpha x) = \alpha T(x)$

A nonlinear transformation *is a mapping that is not linear.*

Theorem 2 (Superposition Principle) $T \in L(X, Y)$ *if and only if*

$$T\left(\sum_{i=1}^{m} \alpha_i x_i\right) = \sum_{i=1}^{m} \alpha_i T\left(x_i\right)$$

Examples of Linear Transformations

Example 1

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T([x, y]) = [2x + 3y, x - y]$
- $T : \mathbb{R}^2 \to \mathbb{R}^1$ *by* $T([x, y]) = [x]$
- $D : \mathbb{P}^n \to \mathbb{P}^{n-1}$ *by* $D(p) = \frac{d}{dx} p(x)$
- $I : \mathcal{C}[0, 1] \to \mathbb{R}$ *by* $I(f) = \int_0^1 f(t)\, dt$
- *Let* $k \in \mathcal{C}[a, b] \times \mathcal{C}[a, b]$ *such that for any* $x \in \mathcal{C}[a, b]$,

$$\widehat{x}(s) = \int_a^b x(t)k(s, t)\, dt \in \mathcal{C}[a, b]$$

Then $\widehat{\cdot} : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ *is a linear transformation.*[a]

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[a] Fredholm Integral Equation of the First Type *or a* kernel transform