# Examples of Linear Transformations

## Example 1

- *Let* $\mathcal{L}_1^c = \{f : \mathbb{R} \to \mathbb{C} \mid f \in \mathcal{C}(\mathbb{R})$ *and* $\int_{\mathbb{R}} |f| < \infty\}$. *Now define the* Fourier transform $\mathcal{F}(f) \in \mathcal{L}_1^c$ *by*

$$\mathcal{F}(f)(s) = \int_{\mathbb{R}} f(t)\, e^{-ist}\, dt$$

  *Then* $\mathcal{F} : \mathcal{L}_1^c \to \mathcal{L}_1^c$ *is a linear transformation.*

- *Let* $z \in \mathbb{C}$. *Then* $\bar{z} =$ *(the complex conjugate of* $z$*) is a* nonlinear transformation*.*

- *Let* $|\cdot|$ *be the absolute value function on* $\mathbb{R}$. *Is* $|\cdot|$ *a linear transformation from* $\mathbb{R}$ *to* $\mathbb{R}$*?*

# Null Space and Range Space

**Definition 1** *Let* $T \in L(X, Y)$*. Then the*

1. null space $\mathcal{N}(T)$ *(or* kernel $\ker(T)$*) is the set*

$$\mathcal{N}(T) = \{x \in X \mid T(x) = 0\},$$

2. range space $\mathcal{R}(T)$ *(or* image space*) is the set*

$$\mathcal{R}(T) = \{y \in Y \mid y = T(x) \text{ for some } x \in X\} = T(X).$$

**Theorem 1** *Let* $T \in L(X, Y)$*. Then*

1. $\mathcal{N}(T)$ *is a subspace of* $X$,
2. $\mathcal{R}(T)$ *is a subspace of* $Y$.

**Pf**. Exercise (3.4.20)

# Range & Dimension

**Theorem 2** *If* $T \in L(X, Y)$, *then* $\dim(\mathcal{R}(T)) \leq \dim(X)$*.*

**Pf**. Assume $X \neq \{0\} \neq \mathcal{R}(T)$, otherwise the result is trivial. Set $n = dim(X) > 0$. Choose $\{y_1, \ldots, y_{n+1}\} \subseteq \dim(\mathcal{R}(T))$. For each $i$, find $x_i$ such that $T(x_i) = y_i$. Since $\dim(X) = n$, we know that there are scalars $\alpha_i$ so that

$$\alpha_1 x_1 + \cdots + \alpha_{n+1} x_{n+1} = 0$$

Applying $T$ to this linear combination yields

$$\alpha_1 y_1 + \cdots + \alpha_{n+1} y_{n+1} = 0$$

Since the $y_i$ were arbitrary, every subset of $\dim(\mathcal{R}(T))$ with $n + 1$ vectors is lin. dep. Thence $\dim(\mathcal{R}(T)) \leq n$.