The Dimension Theorem

Theorem 1 *The inverse image of a basis under a linear transformation is linearly independent. I.e., Let* $T \in L(X,Y)$ *and let* $\mathcal{B}_Y = \{y_i\}$*. For each* i, *choose an* x_i *such that* $T(x_i) = y_i$*. Then the set* $\{x_i\}$ *is linearly independent.*

Pf. Exercise (3.4.24)

Theorem 2 (The Dimension Theorem) *Let* $T \in L(X,Y)$ *with* $\dim(X) < \infty$*. Then*

$$\dim(\mathcal{R}(T)) + \dim(\mathcal{N}(T)) = \dim(X).$$

Pf. Set $\dim(X) = n$ and $\dim(\mathcal{N}(T)) = s$ and set $r = n - s$.
(*Need to show*: $\dim(\mathcal{R}(T)) = r = n - s$.)

The Dimension Theorem Proof

Pf. Find a basis for $\mathcal{N}(T)$ labeling the vectors $\{e_1, \ldots, e_s\}$. Extend this set to a basis for X by adding r vectors to have $\mathcal{B} = \{x_1, \ldots, x_r, e_1 \ldots, e_s\}$. Since $\mathcal{B}$ is a basis, then $T(\mathcal{B})$ spans $\mathcal{R}(T)$. Since $T(e_i) = 0$, then $T(\{x_1, \ldots, x_r\})$ spans $\mathcal{R}(T)$. Set $y_i = T(x_i)$; so $\{y_1, \ldots, y_r\}$ spans $\mathcal{R}(T)$. Suppose a linear combination $\alpha_1 y_1 + \cdots + \alpha_r y_r = 0$. Then because $\sum_r \alpha_i T(x_i) = T(\sum_r \alpha_i x_i)$, we have that $\sum_r \alpha_i x_i \in \mathcal{N}(T)$, thus $\sum_r \alpha_i x_i = \sum_s \gamma_i e_i$ which can be written as

$$\alpha_1 x_1 + \cdots + \alpha_r x_r - \gamma_1 e_1 - \cdots - \gamma_s e_s = 0$$

which implies each $\alpha_i = 0$. Hence $\dim(\mathcal{R}(T)) = r$.

– *What about the cases* $s = 0$ *and* n? (Group-Project time!)