Rank & Nullity

Definition 1 (Rank and Nullity of a Linear Transformation) *Let* $T \in L(X, Y)$*.*

- *The* rank ρ *of* T *is the dimension of the range space;* $\rho(T) = \dim(\mathfrak{R}(T))$
- *The* nullity ν *of* T *is the dimension of the nullspace;* $\nu(T) = \dim(\mathfrak{N}(T))$

Corollary 1 (Fundamental Theorem of Linear Algebra) *Let* $T \in L(X, Y)$ *where* $\dim(X) = n$*. Then*

$$\rho(T) + \nu(T) = n$$

Pf. $\checkmark$

“Affine Nullspace”

Corollary 2 *Let* $T \in (X, Y)$ *where* $\dim(X) < \infty$, *and let* $\mathcal{B} = \{x_1, \ldots, x_s\}$ *be a basis for* $\mathfrak{N}(T)$ *so that* $\dim(\mathfrak{N}(T) = s$. *Then*

1. *a vector* $x \in X$ *satisfies* $T(x) = 0$ *iff there is a unique set of scalars* α_i s.t. $x = \sum_{i=1}^{s} \alpha_i x_i$,
2. *a vector* $y_0 \in Y$ *is in* $\mathfrak{R}(T)$ *iff there is at least one vector* $x \in X$ s.t. $y_0 = T(x)$,
3. *if vectors* $x_0 \in X$ *and* $y_0 \in Y$ *are s.t.* $T(x_0) = y_0$, *then* $x \in X$ *satisfies* $T(x) = y_0$ *iff there is a unique set of scalars* β_i s.t. $x = x_0 + \sum_{i=1}^{s} \beta_i x_i$.

Pf. ✓

Inverses

Theorem 3 *Let* $T \in L(X, Y)$.

1. T^{-1} *exists iff* $T(x) = 0$ *implies* $x = 0$*; i.e.,* $\mathfrak{N}(T) = \{0\}$.
2. *If* T^{-1} *exists, then* $T^{-1} \in L(\mathfrak{R}(T), X)$.

Pf. $1.$ ($\Leftarrow$) Assume $\mathfrak{N}(T) = \{0\}$. Then $T(x_1) = T(x_2) \Leftrightarrow T(x_1) - T(x_2) = 0 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 - x_2 \in \mathfrak{N}(T) \Leftrightarrow x_1 = x_2$.
($\Rightarrow$) Now assume that T^{-1} exists and that $T(x) = 0$. Since $T(0) = 0$, then $T(x) = T(0)$. Whence $x = 0$.

$2.$ Assume that T is nonsingular and that $T(x_1) = y_1$, $T(x_2) = y_2$. Then $T^{-1}(y_1 + y_2) = T^{-1}(T(x_1) + T(x_2)) = T^{-1}(T(x_1 + x_2)) = x_1 + x_2 = T^{-1}(y_1) + T^{-1}(y_2)$. For $\alpha \in F$, $T^{-1}(\alpha y_1) = T^{-1}(\alpha T(x_1)) = T^{-1}(T(\alpha x_1) = \alpha x_1 = \alpha T^{-1}(y_1)$.

Examples

Example Set 1

- *Let* $T([a,b]) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} [a,b]$. *Show* T *is nonsingular.*
- *Let* $S([a,b]) = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} [a,b]$. *Show* T *is singular.*
- $\mathcal{D} : \mathbb{P} \to \mathbb{P}$ *defined by* $\mathcal{D}(p) = \dfrac{dp}{dx}$ *is singular.*
- *Is* $\mathcal{I} : \mathbb{P} \to \mathbb{P}$ *defined by* $\mathcal{I}(p) = \int p\, dx$ *nonsingular?*
- *Is* $T([a,b]) = [a+b, 0, a-b, 0, 0]$ *invertible?*