Singular and Nonsingular Examples

Example Set 1

- *Let* $T([a,b]) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} [a,b]$. *Show* T *is nonsingular.*
- *Let* $S([a,b]) = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} [a,b]$. *Show* T *is singular.*
- $\mathcal{D} : \mathbb{P} \to \mathbb{P}$ *defined by* $\mathcal{D}(p) = \dfrac{dp}{dx}$ *is singular.*
- *Is* $\mathcal{I} : \mathbb{P} \to \mathbb{P}$ *defined by* $\mathcal{I}(p) = \int p\,dx$ *nonsingular?*
- *Is* $T([a,b]) = [a+b, 0, a-b, 0, 0]$ *invertible?*

“Inverse Results”

Theorem 1 *Let $T \in L(X, Y)$ with $\dim(X) < \infty$. Then T is invertible if and only if $\rho(T) = \dim(X)$. T is said to have “full rank.”*

Pf. $\checkmark$

Theorem 2 *Let $T \in L(X, Y)$ with $\dim(X) = \dim(Y) = n$ where $n < \infty$. Then T is invertible if and only if $\mathfrak{R}(T) = Y$.*

Pf. $(\Rightarrow)$ T invertible implies that $\dim(\mathfrak{R}(T)) = n = \dim(Y)$. Since $\mathfrak{R}(T)$ is a subspace of Y, then $\mathfrak{R}(T) = Y$.

$(\Leftarrow)$ Choose a basis $\mathcal{B} = \{y_1, \ldots, y_n\}$ for $\mathfrak{R}(T) = Y$. Then, since $T^{-1}(\mathcal{B})$ is an independent set of size n, it forms a basis for X. Hence the only set of scalars for which $\sum_i \alpha_i x_i = 0$ is $\alpha_i = 0$. Whence $\mathfrak{N}(T) = \{0\}$, so T is invertible.

Collected Results, I

Theorem 3 (Invertible Linear Transformations) *Let X and Y be vector spaces over F and let $T \in L(X, Y)$. TFAE:*

1. *T is invertible or nonsingular*
2. *T is injective or 1–1*
3. *$T(x) = 0$ implies $x = 0$; i.e., $\mathfrak{N}(T) = \{0\}$*
4. *For each $y \in Y$, $\exists$ a unique $x \in X$ such that $T(x) = y$*
5. *If $T(x_1) = T(x_2)$, then $x_1 = x_2$*
6. *If $x_1 \neq x_2$, then $T(x_1) \neq T(x_2)$,*

If X is finite dimensional, then TFAE:

7. *T is injective*
8. *$\rho(T) = \dim(X)$*

Collected Results, II

Theorem 4 (Surjective Linear Transformations) *Let X and Y be vector spaces over F and let $T \in L(X, Y)$. TFAE:*

1. *T is surjective or onto*
2. *For $y \in Y$, there is at least one $x \in X$ such that $T(x) = y$*

If X and Y are finite dimensional, then TFAE:

3. *T is surjective*
4. *$\rho(T) = \dim(Y)$*

Pf. $\checkmark$

Collected Results, III

Theorem 5 (Bijective Linear Transformations) *Let X and Y be vector spaces over F and let $T \in L(X, Y)$. TFAE:*

1. *T is bijective or onto*
2. *For $y \in Y$, there is a unique $x \in X$ such that $T(x) = y$*

If X and Y are finite dimensional, then TFAE:

3. *T is surjective*
4. *$\rho(T) = \dim(X) = \dim(Y)$*

Theorem 6 (Common Finite Dimension) *Let X and Y be vector spaces over F with finite dimension n and $T \in L(X, Y)$. Then*

$$T\text{:injective} \Leftrightarrow T\text{:surjective} \Leftrightarrow T\text{:bijective} \Leftrightarrow T\text{:invertible}$$

Transformation Spaces

Definition 1 *For S and T in $L(X,Y)$ and α in F, define*

1. $S+T$ *by* $(S+T)(x) \triangleq S(x)+T(x)$
2. $\alpha\, S$ *by* $(\alpha\, S)(x) \triangleq \alpha\, S(x)$
3. $S \circ T$ *by* $(S \circ T)(x) \triangleq S\,(T(x))$

Theorem 7 $L(X,Y)$ *is a vector space over F (using 1 & 2)*

Theorem 8 $L(X,X)$ *is an associative algebra with identity over F (using 1, 2, & 3, and identity $I(x)=x$)*