Transformation Spaces

Definition 1 *For S and T in $L(X,Y)$ and α in F, define*

1. $S+T$ *by* $(S+T)(x) \triangleq S(x)+T(x)$
2. $\alpha\, S$ *by* $(\alpha\, S)(x) \triangleq \alpha\, S(x)$
3. ST *by* $(ST)(x) \triangleq S\,(T(x))$ *when* $\mathrm{range}(T) \subseteq \mathrm{dom}(S)$

Theorem 1 *Let $S, T, U \in L(X,X)$. Then*

1. *If $ST = US = I$, then S is bijective and $S^{-1} = T = U$.*
2. *If S is bijective, then $(S^{-1})^{-1} = S$.*
3. *If S and T are bijective, then $(ST)^{-1} = T^{-1}S^{-1}$.*
4. *If S is bijective and $\alpha \neq 0$, then $(\alpha S)^{-1} = (1/\alpha)\cdot S^{-1}$.*

Polynomials of Transforms

Theorem 2 *$L(X, X)$ is an associative algebra[a] with identity over F (using 1, 2, & 3, and identity $I(x) = x$). $L(X, X)$ is usually noncommutative.*

Definition 2 (Powers of Transforms) *Let $T \in L(X, X)$. Then set $T^0 = I$ and, for $n > 0$, define $T^{(n)} \triangleq T \cdot T^{(n-1)}$ and $T^{(-n)} \triangleq \left(T^{-1}\right)^n$.*

Definition 3 *Let $p \in \mathbb{P}^n$, so that $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$. For $T \in L(X, X)$, define*

$$p(T) = a_0 I + a_1 T + \cdots + a_n T^n = \sum_{i=0}^{n} \alpha_i T^i.$$

[a] "Vector space plus multiplication." See pg. 56 and 104 of the text.

Finite Dimension Structure Theorem

Definition 4 *X is* isomorphic *to Y, written $X \cong Y$, if and only if there is a bijection $T \in L(X,Y)$.*

Theorem 3 (Structure Theorem) *Every n-dimensional vector space X over the field F is isomorphic to F^n.*

Pf. Choose a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ for X. Then define $T \in L(X, F^n)$ by

$$T\left(\sum_{i=1}^{n} \alpha_i e_i\right) = [\alpha_1, \alpha_2, \ldots, \alpha_n]$$

Corollary 4 *Let F be a field and n be a positive integer. There is exactly one vector space of dimension n over F.*