Linear Functionals

Definition 1 *Let X be a vector space over F. Then $f \in L(X, F)$ is called a* linear functional*.*

Example Set 1

- *Let $f \in \mathcal{C}[a, b]$. Then $F(f) = \int_a^b f(t)\, dt$ is a linear functional.*
- *Let $f \in \mathcal{C}[a, b]$ and choose $k \in \mathcal{C}[a, b]$. Then $F_k(f) = \int_a^b f(t)k(t)\, dt$ is a linear functional.*
- *Let $f \in \mathcal{C}[a, b]$ and $x_0 \in [a, b]$. Is $\frac{df}{dt}(x_0)$ a linear functional?*
- *Let F be a field. The mappings $\mathrm{proj}_i : F^n \to F$ for $i = 1..n$ given by $\mathrm{proj}_i\left([\alpha_1, \alpha_2, \ldots, \alpha_n]\right) = \alpha_i$ are linear functionals. $\phi = \sum \alpha_i \,\mathrm{proj}_i$ is also a linear functional.*

Vector Space of Linear Functionals

Definition 2 *Let* X *be a vector space over* F. *Define* $X^f = L(X, F)$. *When* $f \in X^f$ *is evaluated at the vector* $x \in X$, *we use the notation* $f(x) \stackrel{\Delta}{=} \langle x, f \rangle$. *Using* x' *in place of* $f \in X^f$, *we see*

$$
\begin{aligned}
(f_1 + f_2)(x) &= \langle x, x_1' + x_2' \rangle \stackrel{\Delta}{=} \langle x, x_1' \rangle + \langle x, x_2' \rangle \\
&= f_1(x) + f_2(x)
\end{aligned}
$$

and

$$
\begin{aligned}
(\alpha f)(x) &= \langle x, \alpha x' \rangle \stackrel{\Delta}{=} \alpha \langle x, x' \rangle \\
&= \alpha f(x)
\end{aligned}
$$

Theorem 1 $X^f = L(X, F)$ *is a vector space over* F *called the* algebraic conjugate *of* X.

Algebraic Conjugate Basis

Theorem 2 *Let X be a vector space with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ and let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of arbitrarily chosen scalars. Then there is a unique linear functional $x' \in X^f$ such that $\langle e_i, x' \rangle = \alpha_i$ for $i = 1..n$.*

Pf. $(\exists)$ For every $x \in X$, we have unique scalars ξ_i such that $x = \sum_n \xi_i e_i$. Define $x' \in X^f$ by $\langle x, x' \rangle = \sum_n \alpha_i \xi_i$. If $x = e_i$ for some i, then $\xi_i = 1$ and $\xi_j = 0$ for every $j \neq i$. Hence $\langle x, x' \rangle = \alpha_i$; i.e., $\langle e_i, x' \rangle = \alpha_i$.
$(!)$ Suppose $\langle e_i, x_1' \rangle = \alpha_i$ and $\langle e_i, x_2' \rangle = \alpha_i$ for $i = 1..n$. Then $\langle e_i, x_1' \rangle - \langle e_i, x_2' \rangle = 0$ for $i = 1..n$, and so $\langle e_i, x_1' - x_2' \rangle = 0$ for $i = 1..n$. This implies that $x_1' = x_2'$.

Definition 3 (Kronecker Delta) *Set* $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

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