Conjugate Dimension Theorem

Theorem 1 *Let* X *be a finite dimensional vector space with basis* $\mathcal{B} = \{e_1, \ldots, e_n\}$. *Then there exists a unique basis* $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ *for* X^f *such that* $\langle e_i, e'_j \rangle = \delta_{ij}$; *we call* $\mathcal{B}'$ *the* dual basis *of* $\mathcal{B}$. *Further* $\dim(X) = n = \dim(X^f)$.

Pf. There exists a unique set of linear functionals $\mathcal{B}' = \{e'_j\}$ such that $\langle e_i, e'_j \rangle = \delta_{ij}$ for $i, j = 1..n$ which are found by applying Thm **??** to the sets $A_j = \{\delta_{ij} | j = 1..n\}$.

$(\mathcal{B}'$ is linearly independent$)$ Since $\sum \beta_i e'_i = 0$ implies

$$0 = \left\langle e_j, \sum_i \beta_i e'_i \right\rangle = \sum_i \beta_i \langle e_j, e'_i \rangle = \sum_i \beta_i \delta_{ij} = \beta_j$$

Conjugate Dimension Theorem, II

(Pf.) $(\mathcal{B}'$ spans $X^f)$ Let $x' \in X^f$ and define $\alpha_i = \langle e_i, x' \rangle$. (This form is often called a *projection*.) For $x \in X$, there are scalars so that $x = \sum_i \xi_i e_i$. Then

$$\langle x, x' \rangle = \left\langle \sum_i \xi_i e_i, x' \right\rangle = \sum_i \langle \xi_i e_i, x' \rangle = \sum_i \xi_i \langle e_i, x' \rangle = \sum_i \xi_i \alpha_i$$

It also follows that $\langle x, e'_j \rangle = \sum_i \xi_i \langle e_i, e'_j \rangle = \xi_j$. Combine these two results to obtain

$$\langle x, x' \rangle = \sum_i \alpha_i \langle x, e'_i \rangle = \left\langle x, \sum_i \alpha_i e'_i \right\rangle$$

which gives us $x' = \sum_i \alpha_i e'_i$.

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