Algebraic Transpose

Definition 1 (Algebraic Transpose) *Let* $S \in L(X, Y)$. *Then* $S^T : Y^f \to X^F$ *given by* $\langle x, S^T y' \rangle = \langle Sx, y' \rangle$ *is the* algebraic transpose *of* S.

Example 1 *Let* $X = \mathbb{R}^3$ *and* $Y = \mathbb{R}^2$. *Define* $S \in L(\mathbb{R}^3, \mathbb{R}^2)$ *by* $y = S(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *and* $y' \in Y^f$ *by* $\langle y, y' \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. *Then* $x' = S^T(y')$ *is found by*

$\langle x, x' \rangle = \langle x, S^T y' \rangle = \langle Sx, y' \rangle$	$x'(x) = (S^T(y'))(x)$
$\langle x, x' \rangle = \langle \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y' \rangle$	$x'(x) = y'(S(x))$
$\langle x, x' \rangle = \langle \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}, y' \rangle$	$x'(x) = y'(\begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix})$
$\langle x, x' \rangle = x_1 + 2x_2 + x_3$	$x'(x) = x_1 + 2x_2 + x_3$

Hence $S^T(y_1') = x'$ *with* $x'(x) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

The Space of Algebraic Transposes

Theorem 1 *Let* S^T *be the algebraic transpose of* S *where* $S \in L(X, Y)$. *Then* $S^T \in L(Y^f, X^f)$.

Pf. (Calculation.)

1. $S^T(y_1' + y_2') = S^T(y_1') + S^T(y_2')$:

$$\langle x, S^T(y_1' + y_2')\rangle = \langle Sx, (y_1' + y_2')\rangle = \langle Sx, y_1'\rangle + \langle Sx, y_2'\rangle$$
$$= \langle x, S^T y_1'\rangle + \langle x, S^T y_2'\rangle$$

2. $S^T(\alpha y') = \alpha\, S^T(y')$:

$$\langle x, S^T(\alpha y')\rangle = \langle Sx, \alpha y'\rangle = \alpha\langle Sx, y'\rangle$$
$$= \alpha\langle x, S^T(y')\rangle = \langle x, \alpha\, S^T(y')\rangle$$

Algebra of Algebraic Transposes

Theorem 2 *Let I be the identity transform of $L(X,X)$. Then I^T is the identity transform of $L(X^f, X^f)$.*

Theorem 3 *Let 0 be the zero transform of $L(X,Y)$. Then 0^T is the zero transform of $L(Y^f, X^f)$.*

Theorem 4 *Let $R, S \in L(X,Y)$ and $T \in L(Y,Z)$ and let R^T, S^T, and T^T be the respective transposes. Then*

1. $(R+S)^T = R^T + S^T$
2. $(TS)^T = S^T T^T$

Exercise 3.52.32 *(Pg. 113.) Prove the theorems.*

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