21. Bilinear Functionals

Recall: We have $\overline{a+bi} = a - bi$ for any complex number.

Definition 32 (Conjugate Functional) *Let X be a vector space over $\mathbb{C}$. A mapping $g : X \to \mathbb{C}$ is a* conjugate functional *iff $g(\alpha_1 x_1 + \alpha_2 x_2) = \overline{\alpha_1}\, g(x_1) + \overline{\alpha_2}\, g(x_2)$ for all $x_i \in X$ and $\alpha_i \in \mathbb{C}$*

Definition 33 (Bilinear Form) *Let X be a vector space over $\mathbb{C}$. A mapping $g : X \times X \to \mathbb{C}$ is a* bilinear form *or* bilinear functional *iff for all x, x_i and $y, y_i \in X$ and $\alpha_i, \beta_i \in \mathbb{C}$*

1. $g(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1\, g(x_1, y) + \alpha_2\, g(x_2, y)$
2. $g(x, \beta_1 y_1 + \beta_2 y_2) = \overline{\beta_1}\, g(x, y_1) + \overline{\beta_2}\, g(x, y_2)$

That is, g is linear in the first variable *and* conjugate linear in the second variable.

Examples

Example Set 20

1. *Let $X = \mathbb{C}^2$ and g be given by*

$$g(z_1, z_2) = \mathfrak{Re}(z_1)\mathfrak{Re}(z_2) + \mathfrak{Im}(z_1)\mathfrak{Im}(z_2).$$

2. *Let $X = \mathbb{R}^2$ and h be given by*

$$h(x, y) = \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2.$$

3. *Let X be a vector space over $\mathbb{C}$ and let $P, Q \in X^f$. Then $k(x_1, x_2) = P(x_1)\overline{Q(x_2)}$ is a bilinear functional.*
4. *The conjugate of a bilinear functional is also a bilinear functional. I.e., $h(x, y) = \overline{g(x, y)}$.*

Definitions

Definition 34 *Let X be a vector space over $\mathbb{C}$ and g be a bilinear functional on X. Then for all $x, y \in X$,*

- g is symmetric *iff* $g(x,y) = \overline{g(y,x)}$.
- g *is* positive *iff* $g(x,x) \geq 0$.
- g *is* strictly positive *iff* $g(x,x) > 0$ *whenever* $x \neq 0$.
- $\tilde{g}(x) = g(x,x)$ *is the* quadratic form induced by g.

Example 21 *For $h : \mathbb{R}^2 \to \mathbb{R}$ of Example Set 20, No 2, the induced quadratic form is $\tilde{h}(x) = \tilde{h}([x_1, x_2]) = x_1^2 + x_2^2$.*

Theorem 48 *Let g be a bilinear functional. Then*

$$\frac{g(x,y) + g(y,x)}{2} = \tilde{g}\left(\frac{x+y}{2}\right) - \tilde{g}\left(\frac{x-y}{2}\right)$$

Definitions, II

Definition 35 (Polarization) *Let X be a vector space over $\mathbb{C}$ and g be a bilinear functional on X. Then*

$$g(x,y) = \left[\tilde{g}\left(\frac{1}{2}(x+y)\right) - \tilde{g}\left(\frac{1}{2}(x-y)\right)\right] + i\left[\tilde{g}\left(\frac{1}{2}(x+iy)\right) - \tilde{g}\left(\frac{1}{2}(x-iy)\right)\right]$$

Theorem 49 *Let g and h be bilinear functionals on the complex vector space X. If $\tilde{g} = \tilde{h}$, then $g = h$.*

Theorem 50 *A bilinear functional g on a complex vector space X is symmetric iff $\tilde{g}$ is real.*

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