22. Quadratic Forms & Inner Products

Theorem 48 *Let g be a bilinear functional. Then*

$$\frac{g(x,y)+g(y,x)}{2} = \tilde{g}\left(\frac{x+y}{2}\right) - \tilde{g}\left(\frac{x-y}{2}\right)$$

Theorem 49 (Polarization) *Let X be a vector space over $\mathbb{C}$ and g be a bilinear functional on X. Then*

$$g(x,y) = \left[\tilde{g}\left(\frac{x+y}{2}\right) - \tilde{g}\left(\frac{x-y}{2}\right)\right] + i\left[\tilde{g}\left(\frac{x+iy}{2}\right) - \tilde{g}\left(\frac{x-iy}{2}\right)\right]$$

Pfs. ✓

"Symmetry is Real"

Theorem 50 *Let g and h be bilinear functionals on the complex vector space X. If $\tilde{g} = \tilde{h}$, then $g = h$.*

Theorem 51 *A bilinear functional g on a complex vector space X is symmetric iff $\tilde{g}$ is real.*

Pf. ($\Rightarrow$) Let g be symmetric, then $g(x,y) = \overline{g(y,x)}$ so that $\tilde{g}(x) = \overline{\tilde{g}(x)}$. Hence $\tilde{g}$ is real.[a]
($\Leftarrow$) If $\tilde{g}$ is real, set $h(x,y) = \overline{g(y,x)}$. Then $\tilde{h}(x) = \overline{\tilde{g}(x,x)} = g(x,x) = \tilde{g}(x)$; i.e., $\tilde{h} = \tilde{g}$. By the previous theorem, $h = g$, and hence $g(x,y) = \overline{g(y,x)}$. That is g is symmetric.

[a] $z = \bar{z} \Rightarrow x + iy = x - iy \Rightarrow y = 0 \Rightarrow z \in \mathbb{R}$.

Inner Product

Ex. Work through example 3.6.18 on pg. 117.

Definition 32 (Inner Product) *A bilinear functional g is an* inner product *iff*

1. *g is strictly positive* $\quad g(x,x) > 0$ *whenever* $x \neq 0$
2. *g is symmetric* $\quad g(x,y) = \overline{g(y,x)}$

Definition 33 (Inner Product) *(Alternate Definition) A function $(\cdot,\cdot) : X \times X \to \mathbb{C}$ is an* inner product *iff*

1. $(x,x) > 0$ *whenever* $x \neq 0$ *and* $(0,0) = 0$
2. $(x,y) = \overline{(y,x)}$
3. $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$
4. $(x, \alpha y + \beta z) = \bar{\alpha}(x,y) + \bar{\beta}(x,z)$

Inner Product Space

Definition 34 *A complex vector space with an inner product is an* inner product space*. A subspace of an inner product space with the restricted inner product is an* inner product subspace*.*

Definition 35 *Let X be an inner product space. Two vectors x and y are* orthogonal*, written as $x \perp y$, iff $(x,y) = 0$. If x is orthogonal to every vector in a set $A \subseteq X$, then $x \perp A$.*

Example Set 22

1. *Let $X = \mathbb{R}^2$ and let $(x,y) = x_1y_1 + x_2y_2$. Then $\{X; (\cdot,\cdot)\}$ is a real inner product space.*
2. *Let $X = \mathbb{C}^n$ and let $(u,v) = \sum_n u_i\overline{v_i}$. Then $\{X; (\cdot,\cdot)\}$ is a complex inner product space.*

(Go to TOC)