23. Inner Product Space Examples

Example 1 *Let* $X = \mathcal{C}_{\mathbb{C}}[0,1]$ *and set* $(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt$.

1. $(t^2 + it, 1 - it) = \int_0^1 (t^2 + it)(1 + it)dt = \frac{3}{4}\,i$
2. $(t^2 + it, 36t + (2t - 25)i) = 0$, *thence it follows that* $(t^2 + it) \perp (36t + (2t - 25)i)$.
3. $(e^{2\pi kit}, e^{2\pi nit}) = \int_0^1 e^{2\pi(k-n)it}dt = \frac{i}{2\pi(n-k)}\, e^{-2\pi i(n-k)t}\Big|_0^1$
   So $(e^{2\pi kit}, e^{2\pi nit}) = \delta_{kn}$. *Thus* $\mathcal{E} = \{e^{2\pi nit} : n \in \mathbb{Z}\}$ *forms a set of mutually orthogonal functions.*

Orthogonal Polynomials

Example 2 *Let* $X = \mathcal{C}_{\mathbb{R}}[0, 2\pi]$ *and define the inner product* $(f,g) = \int_0^{2\pi} f(t)g(t)\,dt.$

1. $(t^2+t, 1-t) = \int_0^{2\pi} (t^2+t)(1-t)dt = 2\pi^2(1-2\pi^2)$
2. $(\cos(kt), \cos(nt)) = \int_0^{2\pi} \cos(kt)\cos(nt)dt = \frac{\pi}{2}\,\delta_{kn}$. *So* $\{\cos(nt) : n = 0..\infty\}$ *is a mutually orthogonal set.*
3. *Set* $\cos(t) = x$. *Then* $\cos(nt) = \cos(n \arccos(x))$ *becomes a polynomial in* x. *The inner product becomes*

$$(f,g) = \frac{2}{\pi}\int_{-1}^{+1} f(t)g(t)\frac{1}{\sqrt{1-t^2}}\,dt$$

Orthogonal Polynomials, II

Example 2

(3.) *Set* $T_n(x) = \cos(n \arccos(x))$. *Then* $(T_k, T_n) = \delta_{kn}$, *so that* $\{T_n, n = 0..\infty\}$ *forms an orthogonal set of polynomials. The first few Chebychev polynomials are* $T_0(x) = 1$ *and*

$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

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