Projection "Symmetry"

Definition 37 $P \in L(X, X)$ *is* idempotent *iff* $P^2 = P$.

Theorem 54 *P is projection on X_1 along X_2 iff $(I - P)$ is a projection on X_2 along X_1.*

Corollary 55 *If P is projection, then $X = \mathfrak{R}(P) \oplus \mathfrak{N}(P)$*

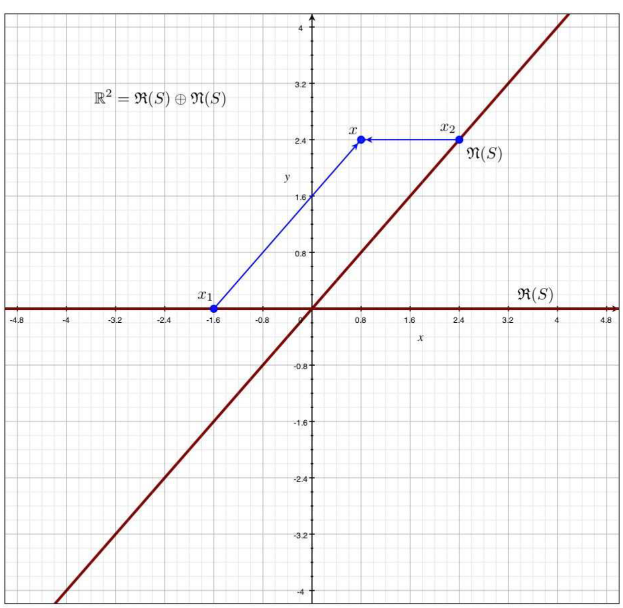
Example Set 27 *Let $X = \mathbb{R}^2$.*

- *Set $R(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$. Is R a projection?*
- *Set $S(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 0 \end{bmatrix}$. Is S a projection?*
- *Set $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_2 \end{bmatrix}$. Is T a projection?*

Definition 38 *P is an* orthogonal projection *on an inner product space iff $\mathfrak{R}(P) \perp \mathfrak{N}(P)$.*

Projection Example S

(Eg.) In $\mathbb{R}^2$, set $S(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 0 \end{bmatrix}$.



(Go to TOC)

24. Projections

Definition 36 *Let* $X = X_1 \oplus X_2$ *and let* $x = x_1 + x_2$ *be the unique representation of* $x \in X$ *relative to* $X_1 \oplus X_2$. *Then define the mapping* P *by* $P(x) = x_1$. *We call* P *the* projection on X_1 along X_2.

Theorem 52 *Let* $X = X_1 \oplus X_2$ *and* P *be the projection on* X_1 *along* X_2. *Then*

1. $P \in L(X, X)$ *and* $P \in L(X, X_1)$
2. $\mathfrak{R}(P) = X_1$
3. $\mathfrak{N}(P) = X_2$

Pf. ✓

Example 25 *Let* $X = \mathbb{R}^2$ *and* $P([x_1, x_2]) = x_2$.

Projections, II

Example 26 *Let* $X = \mathbb{P}^3$ *and* $P(\sum_{i=0}^{3} \alpha_i x^i) = \alpha_0 + \alpha_2 x^2$.

Theorem 53 *Let* $P \in L(X, X)$. *Then* P *is a projection on* $\mathfrak{R}(P)$ *along* $\mathfrak{N}(P)$ *iff* $P^2 = P$.

Pf. $(\Rightarrow)$ Suppose that P is the projection on $\mathfrak{R}(P)$ along $\mathfrak{N}(P)$. Then $X = \mathfrak{R}(P) \oplus \mathfrak{N}(P)$. Let $x = x_1 + x_2$. Then $P^2(x) = P(P(x_1 + x_2)) = P(x_1) = x_1$ Hence $P^2 = P$.
$(\Leftarrow)$ Now suppose that $P^2 = P$. (i) Let $y \in \mathfrak{R}(P)$. Then $\exists x \in X$ so that $P(x) = y$. Whence $P(P(x)) = P(y)$. But $P^2 = P$, so $P(P(x)) = P(x) = y$; i.e. $P(y) = y$. If y is also in $\mathfrak{N}(P)$, then $P(y) = 0$ which implies that $y = 0$. Hence $\mathfrak{R}(P) \cap \mathfrak{N}(P) = \{0\}$. (ii) For $x \in X$, $x = P(x) + (I - P)(x)$. Set $x_1 = P(x)$ and $x_2 = (I - P)(x) = x - x_1$. Thence X is equal to $X_1 \oplus X_2$ with P being the projection on X_1 along X_2.