# 26. Vector Spaces & Matrices

**Background**. Let $X$ be a vector space over the field $F$ and let $\dim(X) = n < \infty$.

- Then $X \cong F^n$
- Then, given a basis $\mathcal{B}_X$, each $x \in X$ can be written as $x = [\alpha_1, \ldots, \alpha_n]_{\mathcal{B}_X}$ in "basis order" (row or col format [a])
- Let $T \in L(X, Y)$. $T$'s action on $\mathcal{B}_X$, i.e., the set $T(\mathcal{B}_X)$, completely determines $T(x)$ for any $x \in X$.
- Let $\mathcal{B}_Y$ be a basis for $Y$. Then there is a *matrix* $\mathbf{T}$ based on $\mathcal{B}_X$ and $\mathcal{B}_Y$, so that

$$y = T(x) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \times \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

[a] "Column" and "row" vectors are artifices to aid the arithmetic.



# Examples and $\mathbf{T} = [T]$

**Example Set 28**

1. *Let* $X = \mathbb{R}^2$. *Then* $[1, 2]_{\{[1,0],[0,1]\}} = [-1, 1]_{\{[1,1],[2,3]\}}$.
2. *Let* $X = \mathbb{P}^2$ *with the "standard basis"* $\{e_i = t^i\}$.
   $x = [1, 2, 3] = 1 + 2t + 3t^2$
3. *Let* $D : \mathbb{P}^3 \to \mathbb{P}^3$ *be differentiation. Then, with the standard basis* $\{1, t, t^2, t^3\}$,

$$D\left(\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} \beta \\ 2\gamma \\ 3\delta \\ 0 \end{bmatrix}$$

**Definition 40 (The matrix of $T$)** *If* $\mathcal{B}_X = \{e_j\}$ *and* $\mathcal{B}_Y = \{f_i\}$, *then* $\mathbf{T} = [a_{ij}]$ *where* $a_{ij} = \mathrm{proj}_i(T(e_j))$ *with* $\mathrm{proj}_i : Y \to Y$ *being the projection on the* $i$*th coordinate of* $Y$ *w.r.t.* $\mathcal{B}_Y$.

# The Matrix of $T$

**Example Set 29** *Let* $T \in L(\mathbb{R}^2, \mathbb{R}^2)$ *given by* $T([x_1, x_2]) = [x_1 - x_2, x_1 + x_2]$.

1. *Use the standard basis for both.Then* $\mathbf{T} = \begin{bmatrix} +1 & -1 \\ +1 & +1 \end{bmatrix}$.
2. *Use* $\mathcal{B}_X = \{[1,1],[2,3]\}$ *and* $\mathcal{B}_Y = \{[1,2],[4,3]\}$. *Then* $\mathbf{T} = \begin{bmatrix} \frac{12}{5} & -\frac{2}{5} \\ \frac{37}{5} & -\frac{2}{5} \end{bmatrix}$.

*—We now return you to the regularly scheduled program.—*

**Definition 41 (Coordinate representation)** *Let* $x \in X$ *and let* $\mathcal{B} = \{e_1, \ldots, e_n\}$ *be a basis for* $X$. *Then there are unique scalars* $\xi_j$ *such that* $x = \sum_j \xi_j$. *Write* $x$ *in* coordinate representation *with respect to the basis* $\mathcal{B}$ *as* $x = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}_{\mathcal{B}}$.



# The Transition Matrix

**Example 30** *Let* $\mathcal{B} = \{e_1, \ldots, e_4\}$ *be the standard basis for* $\mathbb{R}^4$. *Set* $\mathcal{B}^* = \{[1,2,1,0],[3,3,3,0],[2,-10,0,0],[-2,1,-6,2]\}$. *Then, for* $x \in \mathbb{R}^4$, *define* $T_{\mathcal{B}^* \to \mathcal{B}}$ *by* $\begin{bmatrix} e_1 & \ldots & e_4 \end{bmatrix}$ *so*

$$[x]_{\mathcal{B}} = \begin{bmatrix} 1 & 3 & 2 & -2 \\ 2 & 3 & -10 & 1 \\ 1 & 3 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix} \times [x]_{\mathcal{B}^*}$$

*Hence*

$$[x]_{\mathcal{B}^*} = \begin{bmatrix} 1 & 3 & 2 & -2 \\ 2 & 3 & -10 & 1 \\ 1 & 3 & 0 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \times [x]_{\mathcal{B}} = \begin{bmatrix} 5 & 1 & -6 & -\frac{27}{2} \\ -\frac{5}{3} & -\frac{1}{3} & \frac{7}{3} & \frac{11}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \times [x]_{\mathcal{B}}$$

**Query**: How is a vector in $\mathcal{B}_1$ coordinates expressed in $\mathcal{B}_2$ coordinates? Can of cake: use

$$T_{\mathcal{B}_1 \to \mathcal{B}_2} = T_{\mathcal{B}_2 \to \mathcal{B}}^{-1} \times T_{\mathcal{B}_1 \to \mathcal{B}}$$

(Go to TOC)