27. Rank of a Matrix

Theorem 57 *Let $T \in L(X, Y)$ where $\dim(X) = n$ and $\dim(Y) = m$. The $\rho(T) = r$ iff there are bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ such that*

$$\mathbf{T} = \left.\underbrace{\begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \cdots & \cdots & \\ 0 & \dots & 0 & 1 \end{matrix}}^{r} & \begin{matrix} 0 & \dots & 0 \\ 0 & & \\ & \cdots & \\ 0 & \dots & 0 \end{matrix} \\ \begin{matrix} 0 & \dots & 0 \\ 0 & & \\ & \cdots & \\ 0 & \dots & 0 \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ 0 & & \\ & \cdots & \\ 0 & \dots & 0 \end{matrix} \end{bmatrix}}_{n=\dim(X)}\right\} m = \dim(Y)$$

The Rank Theorem Examples

Example Set 31

1. *Consider $T \in L(\mathbb{R}^3 \to \mathbb{R}^2)$. Then $\mathbf{T}$ must have one of the forms below (assuming proper choice of bases):*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Explain why.

2. *Consider $T \in L(\mathbb{R}^3 \to \mathbb{R}^4)$. Then $\mathbf{T}$ must have one of the forms below (assuming proper choice of bases):*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Explain why.

The Rank Theorem Proof

Theorem 57 *Let* $T \in L(X, Y)$. *Then* $\rho(T) = r$ *iff* $\mathbf{T} = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix}$.

Pf. $(\Leftrightarrow)$ Let $r = \rho(T)$. Choose a basis for $\mathfrak{N}(T)$ of $n - r$ vectors listing it as $\{e_{r+1}, e_{r+2}, \ldots, e_n\}$. Extend this basis to all of X as $\mathcal{B}_X = \{e_1, e_2, \ldots, e_r, e_{r+1}, \ldots, e_n\}$. Calculate $\mathcal{F} = \{T(e_i) \,|\, i = 1..r\}$ which forms a basis for $\mathfrak{R}(T)$. (Thm 3.4.25) Extend $\mathcal{F}$ to a basis $\mathcal{B}_Y$ by adding vectors $\{f_{r+1}, \ldots, f_m\}$. (Thm 3.3.44) Then

$$
\begin{aligned}
f_1 = \mathbf{T}e_1 &= (1)f_1 + (0)f_2 + \cdots + (0)f_r + (0)f_{r+1} + \cdots + (0)f_m \\
f_2 = \mathbf{T}e_2 &= (0)f_1 + (1)f_2 + \cdots + (0)f_r + (0)f_{r+1} + \cdots + (0)f_m \\
&\cdots \\
f_r = \mathbf{T}e_r &= (0)f_1 + (0)f_2 + \cdots + (1)f_r + (0)f_{r+1} + \cdots + (0)f_m \\
0 = \mathbf{T}e_{r+1} &= (0)f_1 + (0)f_2 + \cdots + (0)f_r + (0)f_{r+1} + \cdots + (0)f_m \\
&\cdots \\
0 = \mathbf{T}e_n &= (0)f_1 + (0)f_2 + \cdots + (0)f_r + (0)f_{r+1} + \cdots + (0)f_m
\end{aligned}
$$

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