28. Rank & Algebra of Matrices

Definition 42 *Let $\mathbf{A} \in \mathfrak{M}_{mn}$ be the matrix of $A \in L(X, Y)$ w.r.t. the bases $\mathcal{B}_X$ and $\mathcal{B}_Y$. The* rank *of $\mathbf{A}$ is the largest number of linearly independent columns in $\mathbf{A}$.*

Theorem 58 *Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be comparable/conformal matrices and let $\alpha, \beta \in F$. Then*

1. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
2. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
3. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
4. $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
5. $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$
6. $(\alpha\mathbf{A})(\beta\mathbf{B}) = (\alpha\beta)(\mathbf{AB})$
7. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
8. $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Algebra of Matrices

Theorem 59

1. *The zero matrix $\mathbf{0} = [0_{ij}]$ represents the zero transform $0(x) = 0$ for every basis.*
2. *The identity matrix $\mathbf{I} = [\delta_{ij}]$ represents the identity transform $I(x) = x$ for every basis.*
3. *The matrix $\mathbf{A}$ is nonsingular iff the transform A is nonsingular.*
4. *If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1}$ is unique.*
5. *If $\mathbf{A}_n$ and $\mathbf{B}_n$ are nonsingular, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.*
6. $\mathrm{rank}(\mathbf{A}_n) = n$ *if and only if* $(\mathbf{A}_n x = 0 \Leftrightarrow x = 0)$.
7. *For $\mathbf{A} \in \mathfrak{M}_n$, set $\mathbf{A}^m = \underbrace{\mathbf{A} \cdot \mathbf{A} \cdot \cdots \cdot \mathbf{A}}_{m}$ & $\mathbf{A}^{-m} = (\mathbf{A}^{-1})^m$.*

Partitioned Vectors & Matrices

Partitioning a vector or matrix can be very useful and is natural in direct sums. E.g.,

$$\left[\begin{array}{c} x_1 \\ x_2 \\ \hdashline x_3 \end{array}\right], \quad \left[\begin{array}{cc:c} a_{11} & a_{12} & b_{11} \\ a_{21} & a_{22} & b_{21} \\ \hdashline c_{11} & c_{12} & d_{11} \end{array}\right], \quad \left[\begin{array}{c:c:c} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{11} \\ \hdashline \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{21} \end{array}\right]$$

Theorem 60 *Let $P \in L(X, X)$ be a projection and $\dim(X) = n$. Then there is a basis for $X = \mathfrak{R}(P) \oplus \mathfrak{N}(P)$ s.t.*

$$\mathbf{P} = \left[\begin{array}{cccc:ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right]$$

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29. Similarity & Equivalence

Ab hinc: X and Y are vector spaces over F with $\dim X = n$ and $\dim(Y) = m$.

Theorem 61 *Let $\mathcal{B}_X = \{e_1, \ldots, e_n\}$ be a basis for X and let $\mathbf{P} = [p_{ij}]$ be an $n \times n$ matrix. Set $e'_k = \sum_j p_{jk} e_j$. Then $\mathcal{B}'_X = \{e'_1, \ldots, e'_n\}$ is a basis for X iff $\mathbf{P}$ is nonsingular.*

Pf. *Calculation based on the linear independence of $\mathcal{B}_X$.*

Definition 43 *Let $\mathbf{P}$ be the matrix of Thm 61, then $\mathbf{P}$ is* the matrix of $\mathcal{B}'_X$ w.r.t $\mathcal{B}_X$.

Theorem 62 $\mathbf{P}$ *is the matrix of $\mathcal{B}'_X$ w.r.t $\mathcal{B}_X$ iff $\mathbf{P}^{-1}$ is the matrix of $\mathcal{B}_X$* w.r.t $\mathcal{B}'_X$.

Pf. *Exercise*.