30. Equivalence of Transformations

Theorem 65 *Let $A \in L(X, Y)$ where*

- *A has matrices $\mathbf{A}_{\mathcal{B}_X \to \mathcal{B}_Y}$, and $\mathbf{A}'_{\mathcal{B}'_X \to \mathcal{B}'_Y}$, resp.*
- *$\mathbf{P}$ is the matrix of $\mathcal{B}'_X$ w.r.t. $\mathcal{B}_X$ and $\mathbf{Q}$ of $\mathcal{B}'_Y$ w.r.t. $\mathcal{B}_Y$*

Then $\mathbf{A}' = \mathbf{QAP}$.

Pf.

$$Ae'_i = A \cdot \sum_k p_{ki} e_k = \sum_k p_{ki} A e_k = \sum_k p_{ki} \left(\sum_l a_{lk} f_l \right)$$

$$= \sum_k p_{ki} \left(\sum_l a_{lk} \left[\sum_j q_{jl} f'_j \right] \right) = \sum_k \sum_l \sum_j q_{jl} a_{lk} p_{ki} \cdot f'_j$$

Whence

$$a'_{ij} = \sum_l \sum_k q_{il} a_{lk} p_{kj}$$

Definition of Equivalence

Definition 44 *Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{A}'$ are* equivalent *iff there are nonsingular square matrices $\mathbf{P}_n$ and $\mathbf{Q}_m$ such that $\mathbf{A}' = \mathbf{Q_m} \cdot \mathbf{A} \cdot \mathbf{P_n}$. Equivalence is written as $\mathbf{A}' \sim \mathbf{A}$.*

Theorem 66 *Matrix equivalence is an equivalence relation. I.e., $\sim$ is reflexive, symmetric, and transitive.*

Pf. *Exercise.*

Theorem 67 *Let $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{M}_{m,n}$. Then*

1. *$\mathbf{A}$ is equivalent to $\left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right]$ where $r = \operatorname{rank}(A)$.*
2. *$A \sim B$ iff $\operatorname{rank}(A) = \operatorname{rank}(B)$.*

Equivalence Example

Example 32 *Consider* $A \in L(\mathbb{R}^4, \mathbb{R}^5)$.

Suppose $\mathbf{A} = \begin{bmatrix} 7 & -9 & 5 & -4 \\ 7 & 3 & -8 & -5 \\ 4 & 9 & 5 & 6 \\ 11 & 0 & 10 & 2 \\ 0 & 12 & -13 & -1 \end{bmatrix}$ *and* $\mathbf{A}' = \begin{bmatrix} 1 & 13 & 0 & 12 \\ -21 & -31 & 8 & 6 \\ 13 & 14 & -7 & 15 \\ -1 & 21 & 3 & 0 \\ 11 & -46 & -10 & -21 \end{bmatrix}$.

Then $\mathbf{P} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} -1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 1 \end{bmatrix}$.

1. Show that $\mathbf{A}' = \mathbf{QAP}$.
2. Find the matrix $\left[\begin{array}{c:c} I_r & 0 \\ \hdashline 0 & 0 \end{array}\right]$ equivalent to both $\mathbf{A}$ and $\mathbf{A}'$.

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