Eigenvalue & Eigenvector

Definition 1 *Let $A \in L(X, X)$. A scalar λ such that there is a nonzero $x \in X$ for which $Ax = \lambda x$ is an* eigenvalue *and the corresponding x is an* eigenvector*.*

Definition 2 *The polynomial $p(\lambda) = |A - \lambda I|$ is the* characteristic polynomial of A*.*

Theorem 2 (Cayley-Hamilton) *Let $A \in L(X, X)$. Then $p(A) = 0$.*

Definition 3 *Let $A \in L(X, X)$. Then the subspace Y is an* invariant subspace under A *iff $A(Y) \subseteq Y$; i.e., $\forall y \in Y$, we have $Ay \in Y$.*

Definition 4 *Set $\mathfrak{N}_\lambda(A) = \mathfrak{N}_\lambda = \mathfrak{N}(A - \lambda I)$.*

32. Determinants & Invariants

Recall the following

Theorem 1 *Let $A \in L(X, X)$.*

- $|A| \neq 0$ *iff A is nonsingular*
- $|A \cdot B| = |A| \cdot |B|$
- $|A^{-1}| = |A|^{-1}$
- $|\alpha A| = \alpha^n |A|$ *where $n = dim(A)$*
- $|A| = 0$ *iff*
  - *A has a row/column of zeros*
  - *A has two identical rows/columns*
  - *A has a row/column that is a linear combination of other rows/columns*
  - *$A\mathbf{x} = \mathbf{b}$ has nonunique solutions*

Reduced Linear Transformation

Theorem 3 *Let $A \in L(X,X)$. Then X, $\mathfrak{R}(A)$, $\mathfrak{N}(A)$, and $\{0\}$ are all invariant subspaces under A.*

Example 1 *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right]$. Then $\{0\}$. $\mathfrak{N}(A) = \langle \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right] \rangle$, $\mathfrak{R}(A) = \langle \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] \rangle$, and $\mathbb{R}^2$ are all invariant.*

Example 2 *(Exercise.) Let $B : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 0 \end{bmatrix}$. Then $\{0\}$, $\mathfrak{N}(B) = \langle ? \rangle$, $\mathfrak{R}(B) = \langle ? \rangle$, and $\mathbb{R}^3$ are all invariant.*

Theorem 4 *Let λ be an eigenvalue of $A \in L(X,X)$. Then $\mathfrak{N}_\lambda$ is invariant. (Exercise.)*

Definition 5 *Let $X = Y \oplus Z$ be such that both Y and Z are invariant subspaces under $A \in L(X,X)$. Then A is* reduced *by Y and Z and A can take form $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.*

(TOC)