33. Eigenvalues & Diagonalization

Ab hinc: X is an n-dimensional vector space over F.

Theorem 72 *Let $\{\lambda_i \,|\, i = 1..p\}$ be a set of distinct eigenvalues of $A \in L(X, X)$ with corresponding nonzero eigenvectors $\mathcal{E} = \{e'_i \,|\, i = 1..p\}$. Then $\mathcal{E}$ is linearly independent.*

Pf. Assume $\mathcal{E}$ is dependent. Choose the smallest set of vectors from $\mathcal{E}$ such that $0 = \sum_{i=1}^{r} \alpha_i e'_i$ (reordering the $r \le p$ vectors as needed). Then $0 = A(0) = A\left(\sum_{i=1}^{r} \alpha_i e'_i\right)$ which gives $0 = \sum_{i=1}^{r} (\lambda_i \alpha_i e'_i)$ $^{(*)}$. Now $0 = \lambda_r 0 = \lambda_r \sum_{i=1}^{r} \alpha_i e'_i$, or $0 = \sum_{i=1}^{r} \lambda_r \alpha_i e'_i$ $^{(**)}$. Subtract $^{(*)}$ from $^{(**)}$ to obtain $0 = \sum_{i=1}^{r-1} (\lambda_r - \lambda_i) \alpha_i e'_i$ which contradicts r being minimal. Hence $\mathcal{E}$ is linearly independent.

"Eigenbasis"

Theorem 73 *If $A \in L(X, X)$ has n distinct eigenvalues, then there is a basis of eigenvectors $\mathcal{B}_e = \{e_i' \mid i = 1..p\}$ such that the matrix of A is $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.*

Pf. *Exercise.*

Corollary 74 *If $A \in L(X, X)$ has n distinct eigenvalues, then every matrix for A is similar to a diagonal matrix.*

Pf. Collect the eigenvectors $\mathcal{E} = \{e_i' \mid i = 1..n\}$. Set $P = [e_1', \ldots, e_n']$. Then $\mathrm{diag}(\lambda_1, \ldots, \lambda_n) = P^{-1} A\, P$.

Example 35 *See the Maple worksheet. (To see what happens without a "cooked" example, enter the following in Maple:* `with(LinearAlgebra):` `A := RandomMatrix(5,5, generator=rand(-3..3)); Eigenvectors(A);`*)*

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34. "Eigen-Basis" Examples

Example 36

1. *Let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in L(\mathbb{R}^3, \mathbb{R}^3)$. *Then* $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda$ *(and* $m(\lambda) = \lambda^2 - \lambda$*) which indicates that* A *has eigenvalues:* $0, 1, 1$. *The corresponding eigenvectors come from* $\mathfrak{N}_\lambda = \mathfrak{N}(A - \lambda I)$. *So* $\mathfrak{N}_0 = \langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle$ *and* $\mathfrak{N}_1 = \langle \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$. *(Found by solving* $\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ *with* $\lambda = 0$ *and* 1, *respectively.)*
   - (a) *Define* P *and find* P^{-1}.
   - (b) *Calculate the diagonal matrix* $P^{-1} A P$ without *using matrix multiplication.*

“Eigen-Basis” Examples, II

Example 37

1. *Let* $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \in L(\mathbb{R}^3, \mathbb{R}^3)$. *Then* $p(\lambda) = \lambda^3 - 3\lambda^2$ *(and* $m(\lambda) = \lambda^3 - 3\lambda^2$*) which indicates that* B *has eigenvalues:* $0, 0,$ *and* 3. *The corresponding eigenvectors come from* $\mathfrak{N}_\lambda = \mathfrak{N}(B - \lambda I)$. *So* $\mathfrak{N}_0 = \langle \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rangle$ *and* $\mathfrak{N}_3 = \langle \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \rangle$. *(Found by solving* $\begin{bmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ *with* $\lambda = 0,$ *and* 3, *respectively.)*
   - *(a) Explain why* P *(and so* P^{-1}*) doesn't exist.*
   - *(b) Can* B *be diagonalized? Why or why not?*

(Solution.)

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