35. Geometric Multiplicity

Definition 50 *Let* λ *be an eigenvalue of* $A \in L(X, X)$. *Then*

- *the* algebraic multiplicity of λ *is the multiplicity as a root of the characteristic polynomial* $p(\lambda)$;
- *the* geometric multiplicity of λ *is the dimension of the nullspace* $\mathfrak{N}_\lambda = \mathfrak{N}(A - \lambda I)$.

Example 38 *Let* $X = \mathbb{R}^3$. *Each matrix below has characteristic polynomial* $p(\lambda) = -(\lambda - 2)^3$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

alg $= 3$, *geo* $= 3$	*alg* $= 3$, *geo* $= 2$	*alg* $= 3$, *geo* $= 1$
$iev = \{e_1, e_2, e_3\}$	$iev = \{e_1, e_3\}$	$iev = \{e_1\}$

Reduction Partition

Theorem 75 *Let $X = X_1 \oplus X_2$ be a direct sum that reduces $A \in L(X,X)$; i.e., A is invariant on X_1 and X_2. Then there is a basis $\mathcal{B}$ for X such that*

$$A_{\mathcal{B}} = \left[\begin{array}{c:c} A_1 & 0 \\ \hdashline 0 & A_2 \end{array}\right]$$

Theorem 76 *Let $X = X_1 \oplus \cdots \oplus X_p$ be a direct sum that reduces $A \in L(X,X)$; i.e., $A_k = A|_{X_k}$ is invariant on X_k for $k = 1..p$. Then there is a basis $\mathcal{B}$ for X such that*

$$A_{\mathcal{B}} = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_p \end{bmatrix} \quad \textit{and} \quad |A_{\mathcal{B}}| = \prod_{k=1}^{p} |A_k|$$

Minimal Polynomial

Example 39 *If* $A \in L(X,X)$ *has* n *distinct eigenvalues in* F*, then* $X = \mathfrak{N}_{\lambda_1} \oplus \cdots \oplus \mathfrak{N}_{\lambda_n}$ *and* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Definition 51 *Let* $A \in L(X,X)$. *Then there is a monic polynomial* $m(\lambda)$, *the* minimal polynomial, *such that*

- $m(A) = 0$
- *any polynomial* m' *with* $m'(A) = 0$ *has* $\deg(m) \leq \deg(m')$

Example 40 *The three matrices of Example 38 have*

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$m(\lambda) = (\lambda - 2) \quad m(\lambda) = (\lambda - 2)^2 \quad m(\lambda) = (\lambda - 2)^3$$

Properties of the Minimal Polynomial

Theorem 77 *The minimal polynomial* $m(\lambda)$ *is unique.*

Theorem 78 *Let* $q(\lambda)$ *be a polynomial such that* $q(A) = 0$. *Then* $m(\lambda) | q(\lambda)$.

Corollary 79 *The minimal polynomial divides the characteristic polynomial; i.e.,* $m(\lambda) | p(\lambda)$.

Theorem 80 *The characteristic polynomial divides a power of the minimal polynomial:* $p(\lambda) \,\big|\, [m(\lambda)]^n$ *where* $n = \dim(X)$.

Corollary 81 $m(\lambda) \,\big|\, p(\lambda) \,\big|\, [m(\lambda)]^n$.

Proofs. *Exercises*.

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