Finding an "eigen-basis" to diagonalize a matrix

1. Set $A = \begin{bmatrix} 1&0&0\\1&0&0\\0&0&1 \end{bmatrix}$. Find $p(\lambda)$ by evaluating $|A-\lambda I| = \begin{vmatrix} 1-\lambda&0&0\\1&-\lambda&0\\0&0&1-\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 + \lambda$ $= \lambda\,(\lambda-1)^2$. Now the eigenvalues are the roots of $p(\lambda)$, so $\Lambda_A = \{0,1\}$. To find eigenvectors that span $\mathfrak{N}_\lambda$, look at $\mathfrak{N}(A-\lambda I)$.

   (a) First use $\lambda = 0$. Find the nullspace associated with $A - 0I = \begin{bmatrix} 1&0&0\\1&0&0\\0&0&1 \end{bmatrix}$ by solving the system $\begin{bmatrix} 1&0&0\\1&0&0\\0&0&1 \end{bmatrix} \cdot \begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix} = \begin{bmatrix} x_1\\x_1\\x_3 \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$. So $x_1 = x_3 = 0$ and x_2 is arbitrary. This implies that $\mathfrak{N}_0 = \langle \begin{bmatrix} 0\\1\\0 \end{bmatrix} \rangle$.

   (b) Now use $\lambda = 1$. Find the nullspace of $A - 1I = \begin{bmatrix} 0&0&0\\1&-1&0\\0&0&0 \end{bmatrix}$ by solving the system $\begin{bmatrix} 0&0&0\\1&-1&0\\0&0&0 \end{bmatrix} \cdot \begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix} = \begin{bmatrix} 0\\x_1-x_2\\0 \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$. So $x_1 = x_2$ and x_3 is arbitrary. This implies that $\mathfrak{N}_1 = \langle \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix} \rangle$.

   Hence an eigenbasis for $\mathbb{R}^3$ w.r.t. A is $\left\{ \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1 \end{bmatrix} \right\}$.
   With the vectors in this order, we have $P = \begin{bmatrix} 0&1&0\\1&1&0\\0&0&1 \end{bmatrix}$. Then $P^{-1}A\,P$ must equal $\begin{bmatrix} 0&0&0\\0&1&0\\0&0&1 \end{bmatrix}$.

2. Set $B = \begin{bmatrix} 1&1&2\\1&1&2\\1&0&1 \end{bmatrix}$. Find $p(\lambda)$ by evaluating $|A-\lambda I| = \lambda^3 - 3\lambda^2 = \lambda^2\,(\lambda-3)$. The eigenvalues are the roots of $p(\lambda)$, so $\Lambda_B = \{0,3\}$.

   (a) First use $\lambda = 0$. Find the nullspace associated with $B - 0I = \begin{bmatrix} 1&1&2\\1&1&2\\1&0&1 \end{bmatrix}$ by solving the system $\begin{bmatrix} 1&1&2\\1&1&2\\1&0&1 \end{bmatrix} \cdot \begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix} = \begin{bmatrix} x_1+x_2+2x_3\\x_1+x_2+2x_3\\x_1+x_3 \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$. So $x_2 = -x_3 = x_1$ and x_1 is arbitrary. This implies that $\mathfrak{N}_0 = \langle \begin{bmatrix} 1\\1\\-1 \end{bmatrix} \rangle$.

   (b) Now use $\lambda = 3$. Find the nullspace of $B - 3I = \begin{bmatrix} -2&1&2\\1&-2&2\\1&0&-2 \end{bmatrix}$ by solving the system $\begin{bmatrix} -2&1&2\\1&-2&2\\1&0&-2 \end{bmatrix} \cdot \begin{bmatrix} x_1\\x_2\\x_3 \end{bmatrix} = \begin{bmatrix} -2x_1+x_2+2x_3\\x_1-2x_2+2x_3\\x_1-2x_3 \end{bmatrix} = \begin{bmatrix} 0\\0\\0 \end{bmatrix}$. So $x_1 = x_2 = 2x_3$ and x_3 is arbitrary. This implies that $\mathfrak{N}_3 = \langle \begin{bmatrix} 2\\2\\1 \end{bmatrix} \rangle$.

   We do not have a full set of eigenvectors to make a basis. Therefore, there is no nonsingular matrix P that can be used to give $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&3 \end{bmatrix}$ from $P^{-1}B\,P$. In this case, B is called a *deficient* matrix.

For $A, B \in L(X,X)$ where $\dim(X) = n$, the critical observation is that for A, we have that $\sum_{\lambda\in\Lambda_A} \dim(\mathfrak{N}_\lambda) = n$, but for B, we see that $\sum_{\lambda\in\Lambda_B} \dim(\mathfrak{N}_\lambda) < n$.

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