Homomorphisms

Definition 1 (Group Homomorphism) *Let* $\{X; +_X\}$ *and* $\{Y; +_Y\}$ *be two groups with* $\rho : X \to Y$*. Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

Definition 2 (Ring Homomorphism) *Let* $\{X; +_X, \cdot_X\}$ *and* $\{Y; +_Y, \cdot_Y\}$ *be two rings with* $\rho : X \to Y$*. Then* ρ *is a* homomorpism *iff*

$$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

$$\rho(x_1 \cdot_X x_2) = \rho(x_1) \cdot_Y \rho(x_2)$$

Vector Space Homomorphism

Definition 3 (Linear Transformation) *Let X and Y be vector spaces over the same field F. Then the relation $\rho : X \to Y$ is a* linear transformation *if and only if for every $\alpha \in F$ and $x_1, x_2 \in X$, it follows that:*

(1) $$\rho(x_1 +_X x_2) = \rho(x_1) +_Y \rho(x_2)$$

(2) $$\rho(\alpha \cdot x_1) = \alpha \cdot \rho(x_1)$$

Linear Transformation

(1)

$$\begin{array}{ccc} [x_1, x_2] & \xrightarrow{+} & x_1 + x_2 \\ \rho\downarrow & & \rho\downarrow \\ [\rho(x_1), \rho(x_2)] & \xrightarrow{+} & \rho(x_1 + x_2) = \\ & & \rho(x_1) + \rho(x_2) \end{array}$$

(2)

$$\begin{array}{ccc} [\alpha, x_1] & \xrightarrow{\cdot} & \alpha \cdot x_1 \\ \rho\downarrow & & \rho\downarrow \\ [\alpha, \rho(x_1)] & \xrightarrow{\cdot} & \rho(\alpha \cdot x_1) = \\ & & \alpha \cdot \rho(x_1) \end{array}$$

Subspace of a Vector Space

Definition 4 (Subspace) *Let X be a vector space over F and let $\emptyset \neq V \subseteq X$. Then V is a* subspace *of X iff*

1. *$\forall u, v \in V$, we have $u + v \in V$* *(closed under addition)*
2. *$\forall \alpha \in F$, $\forall u \in V$, we have $\alpha u \in V$* *(closed under scalar mult.)*

Theorem 1 *A subspace V of a vector space X is a vector space.*

Proof. V is closed under vector addition and scalar multiplication by definition. All remaining vector space properties – with the exception of $0 \in V$ – are inherited from X.
Let $v \in V$ (because $V \neq \emptyset$). Since $0 \in F$, then $0v = 0 \in V$. Thus V is a vector space. $\square$
Note. Every vector space has at least 2 subspaces. What are they?