Now let $f(t) \in F[t]$ be of positive degree. If $f(t) = g(t)h(t)$ implies that either $g(t)$ is a scalar or $h(t)$ is a scalar, then $f(t)$ is said to be **irreducible.**

We close the present section with a statement of the **fundamental theorem of algebra.**

2.3.16. Theorem. Let $f(t) \in F[t]$ be a non-zero polynomial. Let R denote the field of real numbers and let C denote the field of complex numbers.

(i) If $F = C$, then $f(t)$ can be written uniquely, except for order, as a product

$$f(t) = c(t - c_1)(t - c_2) \dots (t - c_n),$$

where $c, c_1, \dots, c_n \in C$.

(ii) If $F = R$, then $f(t)$ can be written uniquely, except for order, as a product

$$f(t) = cf_1(t)f_2(t) \dots f_m(t),$$

where $c \in R$ and the $f_1(t), \dots, f_m(t)$ are monic irreducible polynomials of degree one or two.

2.4. REFERENCES AND NOTES

There are many excellent texts on abstract algebra. For an introductory exposition of this subject refer, e.g., to Birkhoff and MacLane [2.1], Hanneken [2.2], Hu [2.3], Jacobson [2.4], and McCoy [2.6]. The books by Birkhoff and MacLane and Jacobson are standard references. The texts by Hu and McCoy are very readable. The excellent presentation by Hanneken is concise, somewhat abstract, yet very readable. Polynomials over a field are treated extensively in these references. For a brief summary of the properties of polynomials over a field, refer also to Lipschutz [2.5].

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[2.3] S. T. Hu, *Elements of Modern Algebra.* San Francisco, Calif.: Holden-Day, Inc., 1965.

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[2.5] S. Lipschutz, *Linear Algebra.* New York: McGraw-Hill Book Company, 1968.

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3

VECTOR SPACES AND LINEAR TRANSFORMATIONS

In Chapter 1 we considered the set-theoretic structure of mathematical systems, and in Chapter 2 we developed to various degrees of complexity the algebraic structure of mathematical systems. One of the mathematical systems introduced in Chapter 2 was the linear or vector space, a concept of great importance in mathematics and applications.

In the present chapter we further examine properties of linear spaces. Then we consider special types of mappings defined on linear spaces, called linear transformations, and establish several important properties of linear transformations.

In the next chapter we will concern ourselves with finite dimensional vector spaces, and we will consider matrices, which are used to represent linear transformations on finite dimensional vector spaces.

3.1. LINEAR SPACES

We begin by restating the definition of linear space.

3.1.1. Definition. Let X be a non-empty set, let F be a field, let "$+$" denote a mapping of $X \times X$ into X, and let "$\cdot$" denote a mapping of $F \times X$ into X. Let the members $x \in X$ be called **vectors,** let the elements $\alpha \in F$ be called **scalars,** let the operation "$+$" defined on X be called **vector addition,**

and let the mapping "·" be called **scalar multiplication** or **multiplication of vectors by scalars**. Then for each $x, y \in X$ there is a unique element, $x + y \in X$, called the **sum of** x **and** y, and for each $x \in X$ and $\alpha \in F$ there is a unique element, $\alpha \cdot x \triangleq \alpha x \in X$, called the **multiple of** x **by** α. We say that the non-empty set X and the field F, along with the two mappings of vector addition and scalar multiplication constitute a **vector space** or a **linear space** if the following axioms are satisfied:

(i) $x + y = y + x$ for every $x, y \in X$;
(ii) $x + (y + z) = (x + y) + z$ for every $x, y, z \in X$;
(iii) there is a unique vector in X, called the **zero vector** or the **null vector** or the **origin**, which is denoted by 0 and which has the property that $0 + x = x$ for all $x \in X$;
(iv) $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in F$ and for all $x, y \in X$;
(v) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in F$ and for all $x \in X$;
(vi) $(\alpha\beta)x = \alpha(\beta x)$ for all $\alpha, \beta \in F$ and for all $x \in X$;
(vii) $0x = 0$ for all $x \in X$; and
(viii) $1x = x$ for all $x \in X$.

The reader may find it instructive to review the axioms of a field which are summarized in Definition 2.1.63. In (v) the "+" on the left-hand side denotes the operation of addition on F; the "+" on the right-hand side denotes vector addition. Also, in (vi) $\alpha\beta \triangleq \alpha \cdot \beta$, where "·" denotes the operation of mulitplication on F. In (vii) the symbol 0 on the left-hand side is a scalar; the same symbol on the right-hand side denotes a vector. The 1 on the left-hand side of (viii) is the identity element of F relative to "·".

To indicate the relationship between the set of vectors X and the underlying field F, we sometimes refer to a **vector space** X **over field** F. However, usually we speak of a vector space X without making explicit reference to the field F and to the operations of vector addition and scalar multiplication. If F is the field of real numbers we call our vector space a **real vector space**. Similarly, if F is the field of complex numbers, we speak of a **complex vector space**. Throughout this chapter we will usually use lower case Latin letters (e.g., x, y, z) to denote vectors (i.e., elements of X) and lower case Greek letters (e.g., α, β, γ) to denote scalars (i.e., elements of F).

If we agree to denote the element $(-1)x \in X$ simply by $-x$, i.e., $(-1)x \triangleq -x$, then we have $x - x = 1x + (-1)x = (1 - 1)x = 0x = 0$. Thus, if X is a vector space, then for every $x \in X$ there is a unique vector, denoted $-x$, such that $x - x = 0$. There are several other elementary properties of vector spaces which are a direct consequence of the above axioms. Some of these are summarized below. The reader will have no difficulties in verifying these.

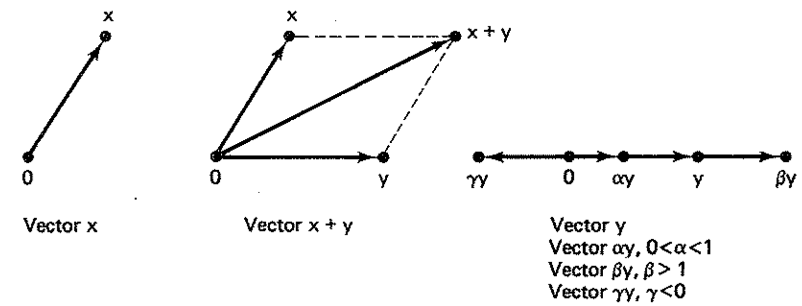
3.1.2. Theorem. Let X be a vector space. If x, y, z are elements in X and if α, β are any members of F, then the following hold:

(i) if $\alpha x = \alpha y$ and $\alpha \neq 0$, then $x = y$;
(ii) If $\alpha x = \beta x$ and $x \neq 0$, then $\alpha = \beta$;
(iii) if $x + y = x + z$, then $y = z$;
(iv) $\alpha 0 = 0$;
(v) $\alpha(x - y) = \alpha x - \alpha y$;
(vi) $(\alpha - \beta)x = \alpha x - \beta x$; and
(vii) $x + y = 0$ implies that $x = -y$.

3.1.3. Exercise. Prove Theorem 3.1.2.

We now consider several important examples of vector spaces.

3.1.4. Example. Let X be the set of all "arrows" in the "plane" emanating from a reference point which we call the **origin** or the **zero vector** or the **null vector**, and which we denote by 0. Let F denote the set of real numbers, and let vector addition and scalar multiplication be defined in the usual way, as shown in Figure A.



3.1.5. Figure A

The reader can readily verify that, for the space described above, all the axioms of a linear space are satisfied, and hence X is a vector space. ■

The purpose of the above example is to provide an intuitive idea of a linear space. We will utilize this space occasionally for purposes of motivation in our development. We must point out however that the terms "plane" and "arrows" were not formally defined, and thus the space X was not really properly defined. In the examples which follow, we give a more precise formulation of vector spaces.

3.1.6. Example. Let $X = R$ denote the set of real numbers, and let F also denote the set of real numbers. We define vector addition to be the usual addition of real numbers and multiplication of vectors $x \in R$ by scalars $\alpha \in F$ to be multiplication of real numbers. It is a simple matter to show that this space is a linear space. ■

3.1.7. Example. Let $X = F^n$ denote the set of all ordered n-tuples of elements from field F. Thus, if $x \in F^n$, then $x = (\xi_1, \xi_2, \ldots, \xi_n)$, where $\xi_i \in F$, $i = 1, \ldots, n$. With $x, y \in F^n$ and $\alpha \in F$, let vector addition and scalar multiplication be defined as

$$x + y = (\xi_1, \xi_2, \ldots, \xi_n) + (\eta_1, \eta_2, \ldots, \eta_n)$$
$$\triangleq (\xi_1 + \eta_1, \xi_2 + \eta_2, \ldots, \xi_n + \eta_n) \tag{3.1.8}$$

and

$$\alpha x = \alpha(\xi_1, \xi_2, \ldots, \xi_n) \triangleq (\alpha\xi_1, \alpha\xi_2, \ldots, \alpha\xi_n). \tag{3.1.9}$$

It should be noted that the symbol "+" on the right-hand side of Eq. (3.1.8) denotes addition on the field F, and the symbol "+" on the left-hand side of Eq. (3.1.8) designates vector addition. (See Theorem 2.1.88.)

In the present case the null vector is defined as $0 = (0, 0, \ldots, 0)$ and the vector $-x$ is defined by $-x = -(\xi_1, \xi_2, \ldots, \xi_n) = (-\xi_1, -\xi_2, \ldots, -\xi_n)$. Utilizing the properties of the field F, all axioms of Definition 3.1.1 are readily verified, and F^n is thus a vector space. We call this space the **space F^n of n-tuples of elements of F.** ■

3.1.10. Example. In Example 3.1.7 let $F = R$, the field of real numbers. Then $X = R^n$ denotes the set of all n-tuples of real numbers. We call the vector space R^n the **n-dimensional real coordinate space.** Similarly, in Example 3.1.7 let $F = C$, the field of complex numbers. Then $X = C^n$ designates the set of all n-tuples of complex numbers. The linear space C^n is called the **n-dimensional complex coordinate space**. ■

In the previous example we used the term *dimension.* At a later point in the present chapter the concept of dimension will be defined precisely and some of its properties will be examined in detail.

3.1.11. Example. Let X denote the set of all infinite sequences of real numbers of the form

$$x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots), \tag{3.1.12}$$

let F denote the field of real numbers, let vector addition be defined similarly as in Eq. (3.1.8), and let scalar multiplication be defined similarly as in Eq. (3.1.9). It is again an easy matter to show that this space is a vector space. We point out that this space, which we denote by R^∞, is simply the collection of *all* infinite sequences; i.e., there is no requirement that any type of convergence of the sequence be implied. ■

3.1.13. Example. Let $X = C^\infty$ denote the set of all infinite sequences of complex numbers of the form (3.1.12), let F represent the field of complex numbers, let vector addition be defined similarly as in Eq. (3.1.8), and let scalar multiplication be defined similarly as in Eq. (3.1.9). Then C^∞ is a vector space. ■

3.1.14. Example. Let X denote the set of all sequences of real numbers having only a finite number of non-zero terms. Thus, if $x \in X$, then

$$x = (\xi_1, \xi_2, \ldots, \xi_l, 0, \ldots 0, \ldots) \tag{3.1.15}$$

for some positive integer l. If we define vector addition similarly as in Eq. (3.1.8), if we define scalar multiplication similarly as in Eq. (3.1.9), and if we let F be the field of real numbers, then we can readily show that X is a real vector space. We call this space the **space of finitely non-zero sequences.**

If X denotes the set of all sequences of complex numbers of the form (3.1.15), if vector addition and scalar multiplication are defined similarly as in equations (3.1.8) and (3.1.9), respectively, then X is again a vector space (a complex vector space). ■

3.1.16. Example. Let X be the set of infinite sequences of real numbers of the form (3.1.12), with the property that $\lim_{n\to\infty} \xi_n = 0$. If F is the field of real numbers, if vector addition is defined similarly as in Eq. (3.1.8), and if scalar multiplication is defined similarly as in Eq. (3.1.9), then X is a vector space. This is so because the sum of two sequences which converge to zero also converges to zero, and because the scalar multiple of a sequence converging to zero also converges to zero. ■

3.1.17. Example. Let X be the set of infinite sequences of real numbers of the form (3.1.12) which are bounded. If vector addition and scalar multiplication are again defined similarly as in (3.1.8) and (3.1.9), respectively, and if F denotes the field of real numbers, then X is a vector space. This space is called the **space of bounded real sequences.**

There also exists, of course, a complex counterpart to this space, the **space of bounded complex sequences.** ■

3.1.18. Example. Let X denote the set of infinite sequences of real numbers of the form (3.1.12), with the property that $\sum_{i=1}^{\infty} |\xi_i| < \infty$. Let F be the field of real numbers, let vector addition be defined similarly as in (3.1.8), and let scalar multiplication be defined similarly as in Eq. (3.1.9). Then X is a vector space. ■

3.1.19. Example. Let X be the set of all real-valued continuous functions defined on the interval $[a, b]$. Thus, if $x \in X$, then $x: [a, b] \to R$ is a real, continuous function defined for all $a \leq t \leq b$. We note that $x = y$ if and only if $x(t) = y(t)$ for all $t \in [a, b]$, and that the null vector is the function which is zero for all $t \in [a, b]$. Let F denote the field of real numbers, let $\alpha \in F$, and let vector addition and scalar multiplication be defined pointwise by

$$(x + y)(t) = x(t) + y(t) \text{ for all } t \in [a, b] \tag{3.1.20}$$

and

$$(\alpha x)(t) = \alpha x(t) \text{ for all } t \in [a, b]. \tag{3.1.21}$$

Then clearly $x + y \in X$ whenever $x, y \in X$, $\alpha x \in X$ whenever $\alpha \in F$ and $x \in X$, and all the axioms of a vector space are satisfied. We call this vector space the **space of real-valued continuous functions on** $[a, b]$ and we denote it by $\mathcal{C}[a, b]$. ■

3.1.22. Example. Let X be the set of all real-valued functions defined on the interval $[a, b]$ such that

$$\int_a^b |x(t)| \, dt < \infty,$$

where integration is taken in the Riemann sense. Let F denote the field of real numbers, and let vector addition and scalar multiplication be defined as in equations (3.1.20) and (3.1.21), respectively. We can readily verify that X is a vector space. ■

3.1.23. Example. Let X denote the set of all real-valued polynomials defined on the interval $[a, b]$, let F be the field of real numbers, and let vector addition and scalar multiplication be defined as in equations (3.1.20) and (3.1.21), respectively. We note that the null vector is the function which is zero for all $t \in [a, b]$, and also, if $x(t)$ is a polynomial, then so is $-x(t)$. Furthermore, we observe that the sum of two polynomials is again a polynomial, and that a scalar multiple of a polynomial is also a polynomial. We can now readily verify that X is a linear space. ■

3.1.24. Example. Let X denote the set of real numbers between $-a < 0$ and $+a > 0$; i.e., if $x \in X$ then $x \in [-a, a]$. Let F be the field of real numbers. Let vector addition and scalar multiplication be as defined in Example 3.1.6. Now, if $\alpha \in F$ is such that $\alpha > 1$, then $\alpha a > a$ and $\alpha a \notin X$. From this it follows that X is *not* a vector space. ■

Vector spaces such as those encountered in Examples 3.1.19, 3.1.22, and 3.1.23 are called **function spaces**. In Chapter 6 we will consider some additional linear spaces.

3.1.25. Exercise. Verify the assertions made in Examples 3.1.6, 3.1.7, 3.1.10, 3.1.11, 3.1.13, 3.1.14, 3.1.16, 3.1.17, 3.1.18, 3.1.19, 3.1.22, and 3.1.23.

3.2. LINEAR SUBSPACES AND DIRECT SUMS

We first introduce the notion of linear subspace. (See also Definition 2.1.102.)

3.2.1. Definition. A non-empty subset Y of a vector space X is called a **linear manifold** or a **linear subspace** in X if (i) $x + y$ is in Y whenever x and y are in Y, and (ii) αx is in Y whenever $\alpha \in F$ and $x \in Y$.

It is an easy matter to verify that a linear manifold Y satisfies all the axioms of a vector space and may as such be regarded as a linear space itself.

3.2.2. Example. The set consisting of the null vector 0 is a linear subspace; i.e., the set $Y = \{0\}$ is a linear subspace. Also, the vector space X is a linear subspace of itself. If a linear subspace Y is not all of X, then we say that Y is a **proper subspace** of X. ■

3.2.3. Example. The set of all real-valued polynomials defined on the interval $[a, b]$ (see Example 3.1.23) is a linear subspace of the vector space consisting of all real-valued continuous functions defined on the interval $[a, b]$ (see Example 3.1.19). ■

Concerning linear subspaces we now state and prove the following result.

3.2.4. Theorem. Let Y and Z be linear subspaces of a vector space X. The intersection of Y and Z, $Y \cap Z$, is also a linear subspace of X.

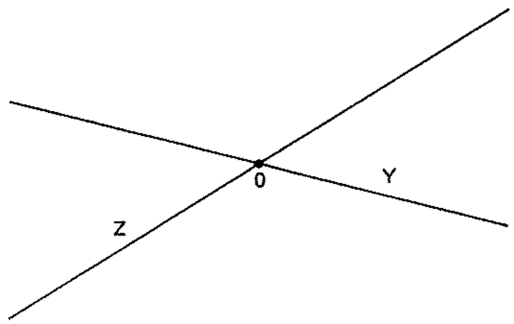
Proof. Since Y and Z are linear subspaces, it follows that $0 \in Y$ and $0 \in Z$, and thus $0 \in Y \cap Z$. Hence, $Y \cap Z$ is non-empty. Now let $\alpha, \beta \in F$, let $x, y \in Y$, and let $x, y \in Z$. Then $\alpha x + \beta y \in Y$ and also $\alpha x + \beta y \in Z$, because Y and Z are both linear subspaces. Hence, $\alpha x + \beta y \in Y \cap Z$ and $Y \cap Z$ is a linear subspace of X. ■

We can extend the above theorem to a more general result.

3.2.5. Theorem. Let X be a vector space and let X_i be a linear subspace of X for every $i \in I$, where I denotes some index set. Then $\bigcap_{i \in I} X_i$ is a linear subspace of X.

3.2.6. Exercise. Prove Theorem 3.2.5.

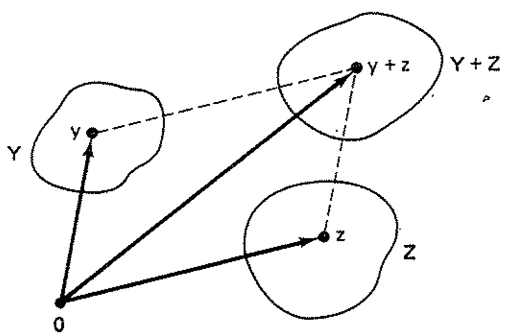
Now consider in the vector space of Example 3.1.4 the subsets Y and Z consisting of two lines intersecting at the origin 0, as shown in Figure B. Clearly, Y and Z are linear subspaces of the vector space X. On the other hand, the union of Y and Z, $Y \cup Z$, obviously does not contain arbitrary sums $\alpha y + \beta z$, where $\alpha, \beta \in F$ and $y \in Y$ and $z \in Z$. From this it follows that if Y and Z are linear subspaces then, in general, the union $Y \cup Z$ is *not* a linear subspace of X.



3.2.7. Figure B

3.2.8. Definition. Let X be a linear space, and let Y and Z be arbitrary subsets of X. The **sum** of sets Y and Z, denoted by $Y + Z$, is the set of all vectors in X which are of the form $y + z$, where $y \in Y$ and $z \in Z$.

The above concept is depicted pictorially in Figure C by utilizing the vector space of Example 3.1.4. With the aid of our next result we can generate various linear subspaces.



3.2.9. Figure C. Sum of two Subsets.

3.2.10. Theorem. Let Y and Z be linear subspaces of a vector space X. Then their sum, $Y + Z$, is also a linear subspace of X.

3.2.11. Exercise. Prove Theorem 3.2.10.

Now let Y and Z be linear subspaces of a vector space X. If $Y \cap Z = \{0\}$, we say that the spaces Y and Z are **disjoint**. We emphasize that this terminology is not consistent with that used in connection with sets. We now have:

3.2.12. Theorem. Let Y and Z be linear subspaces of a vector space X. Then for every $x \in Y + Z$ there exist unique elements $y \in Y$ and $z \in Z$ such that $x = y + z$ if and only if $Y \cap Z = \{0\}$.

Proof. Let $x \in Y + Z$ be such that $x = y_1 + z_1 = y_2 + z_2$, where $y_1, y_2 \in Y$ and where $z_1, z_2 \in Z$. Then clearly $y_1 - y_2 = z_2 - z_1$. Now $y_1 - y_2 \in Y$ and $z_2 - z_1 \in Z$, and since by assumption $Y \cap Z = \{0\}$, it follows that $y_1 - y_2 = 0$ and $z_2 - z_1 = 0$, $y_1 = y_2$ and $z_1 = z_2$. Thus, every $x \in Y + Z$ has a unique representation $x = y + z$, where $y \in Y$ and $z \in Z$, provided that $Y \cap Z = \{0\}$.

Conversely, let us assume that for each $x = y + z \in Y + Z$ the $y \in Y$ and the $z \in Z$ are uniquely determined. Let us further assume that the linear subspaces Y and Z are not disjoint. Then there exists a non-zero vector $v \in Y \cap Z$. In this case we can write $x = y + z = y + z + \alpha v - \alpha v = (y + \alpha v) + (z - \alpha v)$ for all $\alpha \in F$. But this implies that y and z are not unique, which is a contradiction to our hypothesis. Hence, the spaces Y and Z must be disjoint. ■

Theorem 3.2.10 is readily extended to any number of linear subspaces of X. Specifically, if $X_1, \ldots, X_r$ are linear subspaces of X, then $X_1 + \ldots + X_r$ is also a linear subspace of X. This enables us to introduce the following:

3.2.13. Definition. Let $X_1, \ldots, X_r$ be linear subspaces of the vector space X. The sum $X_1 + \ldots + X_r$ is said to be a **direct sum** if for each $x \in X_1 + \ldots + X_r$ there is a unique set of $x_i \in X_i$, $i = 1, \ldots, r$ such that $x = x_1 + \ldots + x_r$. We denote the direct sum of $X_1, \ldots, X_r$ by $X_1 \oplus \ldots \oplus X_r$.

There is a connection between the Cartesian product of two vector spaces and their direct sum. Let Y and Z be two arbitrary linear spaces over the same field F and let $V = Y \times Z$. Thus, if $v \in V$, then v is the ordered pair

$$v = (y, z),$$

where $y \in Y$ and $z \in Z$. Now let us define vector addition as

$$(y_1, z_1) + (y_2, z_2) = (y_1 + y_2, z_1 + z_2) \tag{3.2.14}$$

and scalar multiplication as

$$\alpha(y, z) = (\alpha y, \alpha z), \tag{3.2.15}$$

where $(y_1, z_1), (y_2, z_2) \in V = Y \times Z$ and where $\alpha \in F$. Noting that for each vector $(y, z) \in V$ there is a vector $-(y, z) = (-y, -z) \in V$, and observing that $(0, 0) = (y, z) - (y, z)$ for all elements in V, it is an easy matter to show that the space $V = Y \times Z$ is a linear space. We note that Y is not a linear subspace of V, because, in fact, it is not even a subset of V. However, if we let

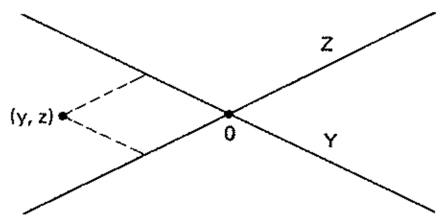
$$Y' = \{(y, 0): y \in Y\},$$

and

$$Z' = \{(0, z): z \in Z\},$$

Then Y' and Z' are linear subspaces of V and $V = Y' \oplus Z'$. By abuse of notation, we frequently express this simply as $V = Y \oplus Z$.

Once more, making use of Example 3.1.4, let Y and Z denote two lines intersecting at the origin 0, as shown in Figure D. The direct sum of linear subspaces Y and Z is in this case the "entire plane."



3.2.16. Figure D

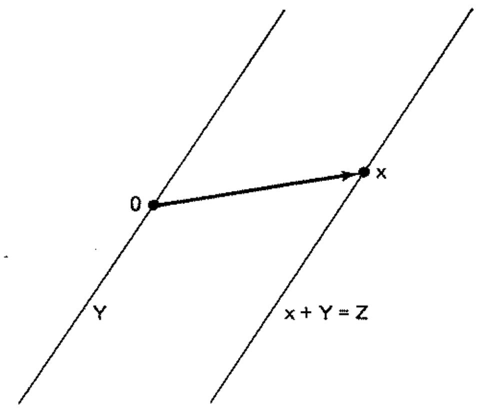
In order that a subset be a linear subspace of a vector space, it is necessary that this subset contain the null vector. Thus, in Figure D, the lines Y and Z passing through the origin 0 are linear subspaces of the plane (see Example 3.1.4). In many applications this requirement is too restrictive and a generalization is called for. We have:

3.2.17. Definition. Let Y be a linear subspace of a vector space X, and let x be a fixed vector in X. We call the translation

$$Z = x + Y \triangleq \{z \in X: z = x + y, y \in Y\}$$

a **linear variety** or a **flat** or an **affine linear subspace** of X.

In Figure E, an example of a linear variety is given for the vector space of Example 3.1.4.



3.2.18. Figure E

3.3. LINEAR INDEPENDENCE, BASES, AND DIMENSION

Throughout the remainder of this and in the following chapter we use the following notation: $\{\alpha_1, \ldots, \alpha_n\}$, $\alpha_i \in F$, denotes an indexed set of scalars, and $\{x_1, \ldots, x_n\}$, $x_i \in X$, denotes an indexed set of vectors.

Before introducing the notions of linear dependence and independence of a set of vectors in a linear space X, we first consider the following.

3.3.1. Definition. Let Y be a set in a linear space X (Y may be a finite set or an infinite set). We say that a vector $x \in X$ is a **finite linear combination of vectors** in Y if there is a finite set of elements $\{y_1, y_2, \ldots, y_n\}$ in Y and a finite set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in F such that

$$x = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_n y_n. \tag{3.3.2}$$

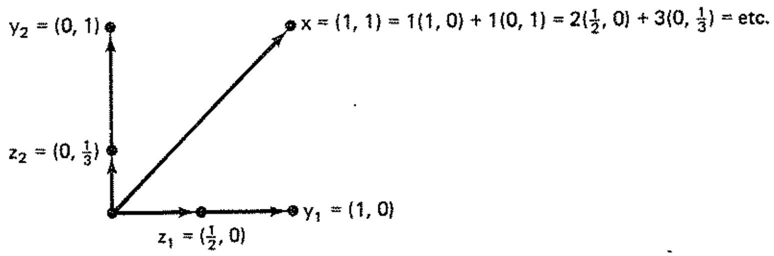
In Eq. (3.3.2) vector addition has been extended in an obvious way from the case of two vectors to the case of n vectors. In later chapters we will consider linear combinations which are not necessarily finite. The representation of x in Eq. (3.3.2) is, of course, not necessarily unique. Thus, in the case of Example 3.1.10, if $X = R^2$ and if $x = (1, 1)$, then x can be represented as

$$x = \alpha_1 y_1 + \alpha_2 y_2 = 1(1, 0) + 1(0, 1)$$

or as

$$x = \beta_1 z_1 + \beta_2 z_2 = 2(\tfrac{1}{2}, 0) + 3(0, \tfrac{1}{3}),$$

etc. This situation is depicted in Figure F.



3.3.3. **Figure F**

3.3.4. Theorem. Let Y be a non-empty subset of a linear space X. Let $V(Y)$ be the set of all finite linear combinations of the vectors from Y; i.e., $y \in V(Y)$ if and only if there is some set of scalars $\{\alpha_1, \ldots, \alpha_m\}$ and some finite subset $\{y_1, \ldots, y_m\}$ of Y such that

$$y = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_m y_m,$$

where m may be any positive integer. Then $V(Y)$ is a linear subspace of X.

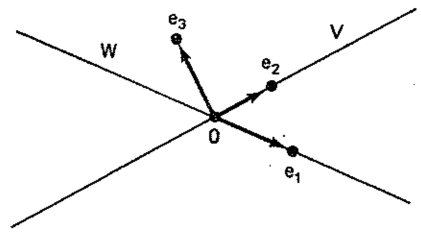
3.3.5. Exercise. Prove Theorem 3.3.4.

Our previous result motivates the following concepts.

3.3.6. Definition. We say the linear space $V(Y)$ in Theorem 3.3.4 is the **linear subspace generated** by the set Y.

3.3.7. Definition. Let Z be a linear subspace of a vector space X. If there exists a set of vectors $Y \subset X$ such that the linear space $V(Y)$ generated by Y is Z, then we say Y **spans** Z.

If, in particular, the space of Example 3.1.4 is considered and if V and W are linear subspaces of X as depicted in Figure G, then the set $Y = \{e_1\}$ spans W, the set $Z = \{e_2\}$ spans V, and the set $M = \{e_1, e_2\}$ spans the vector space X. The set $N = \{e_1, e_2, e_3\}$ also spans the vector space X.



3.3.8. **Figure G.** V and W are Lines Intersecting at Origin O.

3.3.9. Exercise. Show that $V(Y)$ is the smallest linear subspace of a vector space X containing the subset Y of X. Specifically, show that if Z is a linear subspace of X and if Z contains Y, then Z also contains $V(Y)$.

And now the important notion of linear dependence.

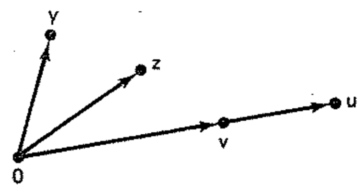
3.3.10. Definition. Let $\{x_1, x_2, \ldots, x_m\}$ be a finite non-empty set in a linear space X. If there exist scalars $\alpha_1, \ldots, \alpha_m \in F$, not all zero, such that

$$\alpha_1 x_1 + \ldots + \alpha_m x_m = 0 \qquad (3.3.11)$$

then the set $\{x_1, x_2, \ldots, x_m\}$ is said to be **linearly dependent.** If a set is not linearly dependent, then it is said to be **linearly independent.** In this case the relation (3.3.11) implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$. An infinite set of vectors Y in X is said to be linearly independent if every finite subset of Y is linearly independent.

Note that the null vector cannot be contained in a set which is linearly independent. Also, if a set of vectors contains a linearly dependent subset, then the whole set is linearly dependent.

If X denotes the space of Example 3.1.4, the set of vectors $\{y, z\}$ in Figure H is linearly independent, while the set of vectors $\{u, v\}$ is linearly dependent.



3.3.12. **Figure H.** Linearly Independent and Linearly Dependent Vectors.

3.3.13. Exercise. Let $X = \mathcal{C}[a, b]$, the set of all real-valued continuous functions on $[a, b]$, where $b > a$. As we saw in Example 3.1.19, this set forms a vector space. Let n be a fixed positive integer, and let us define $x_i \in X$ for $i = 0, 1, 2, \ldots, n$, as follows. For all $t \in [a, b]$, let

$$x_0(t) = 1$$

and

$$x_i(t) = t^i, \quad i = 1, \ldots, n.$$

Let $Y = \{x_0, x_1, \ldots, x_n\}$. Then $V(Y)$ is the set of all polynomials on $[a, b]$ of degree less than or equal to n.

(a) Show that Y is a linearly independent set in X.

(b) Let $X_i = \{x_i\}, i = 0, 1, \ldots, n$; i.e., each X_i is a singleton subset of X. Show that

$$V(Y) = V(X_0) \oplus V(X_1) \oplus \ldots \oplus V(X_n).$$

(c) Let $z_0(t) = 1$ for all $t \in [a, b]$ and let

$$z_k(t) = 1 + t + \ldots + t^k$$

for all $t \in [a, b]$ and $k = 1, \ldots, n$. Show that $Z = \{z_0, z_1, \ldots, z_n\}$ is a linearly independent set in $V(Y)$.

3.3.14. Theorem. Let $\{x_1, x_2, \ldots, x_m\}$ be a linearly independent set in a vector space X. If $\sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1}^{m} \beta_i x_i$, then $\alpha_i = \beta_i$ for all $i = 1, 2, \ldots, m$.

Proof. If $\sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1}^{m} \beta_i x_i$ then $\sum_{i=1}^{m} (\alpha_i - \beta_i) x_i = 0$. Since the set $\{x_1, \ldots, x_m\}$ is linearly independent, we have $(\alpha_i - \beta_i) = 0$ for all $i = 1, \ldots, m$. Therefore $\alpha_i = \beta_i$ for all i. ■

The next result provides us with an alternate way of defining linear dependence.

3.3.15. Theorem. A set of vectors $\{x_1, x_2, \ldots, x_m\}$ in a linear space X is linearly dependent if and only if for some index i, $1 \leq i \leq m$, we can find scalars $\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_m$ such that

$$x_i = \alpha_1 x_1 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \ldots + \alpha_m x_m. \tag{3.3.16}$$

Proof. Assume that Eq. (3.3.16) is satisfied. Then

$$\alpha_1 x_1 + \ldots + \alpha_{i-1} x_{i-1} + (-1) x_i + \alpha_{i+1} x_{i+1} + \ldots + \alpha_m x_m = 0.$$

Thus, $\alpha_i = -1 \neq 0$ is a non-trivial choice of coefficient for which Eq. (3.3.11) holds, and therefore the set $\{x_1, x_2, \ldots, x_m\}$ is linearly dependent.

Conversely, assume that the set $\{x_1, x_2, \ldots, x_m\}$ is linearly dependent. Then there exist coefficients $\alpha_1, \ldots, \alpha_m$ which are not all zero, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m = 0. \tag{3.3.17}$$

Suppose that index i is chosen such that $\alpha_i \neq 0$. Rearranging Eq. (3.3.17) to

$$-\alpha_i x_i = \alpha_1 x_1 + \ldots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \ldots + \alpha_m x_m \tag{3.3.18}$$

and multiplying both sides of Eq. (3.3.18) by $-1/\alpha_i$, we obtain

$$x_i = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{i-1} x_{i-1} + \beta_{i+1} x_{i+1} + \ldots + \beta_m x_m,$$

where $\beta_k = -\alpha_k/\alpha_i$, $k = 1, \ldots, i-1, i+1, \ldots, m$. This concludes our proof. ■

The proof of the next result is left as an exercise.

3.3.19. Theorem. A finite non-empty set Y in a linear space X is linearly indenpendent if and only if for each $y \in V(Y)$, $y \neq 0$, there is a unique finite subset of Y, say $\{x_1, x_2, \ldots, x_m\}$ and a unique set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, such that

$$y = \alpha_1 x_1 + \ldots + \alpha_m x_m.$$

3.3.20. Exercise. Prove Theorem 3.3.19.

3.3.21. Exercise. Let Y be a finite set in a linear space X. Show that Y is linearly independent if and only if there is no proper subset Z of Y such that $V(Z) = V(Y)$.

A concept which is of utmost importance in the study of vector spaces is that of basis of a linear space.

3.3.22. Definition. A set Y in a linear space X is called a **Hamel basis**, or simply a **basis**, for X if

(i) Y is linearly independent; and

(ii) the span of Y is the linear space X itself; i.e., $V(Y) = X$.

As an immediate consequence of this definition we have:

3.3.23. Theorem. Let X be a linear space, and let Y be a linearly independent set in X. Then Y is a basis for $V(Y)$.

3.3.24. Exercise. Prove Theorem 3.3.23.

In order to introduce the notion of dimension of a vector space we show that if a linear space X is generated by a finite number of linearly independent elements, then this number of elements must be unique. We first prove the following result.

3.3.25. Theorem. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for a linear space X. Then for each vector $x \in X$ there exist *unique* scalars $\alpha_1, \ldots, \alpha_n$ such that

$$x = \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

Proof. Since $x_1, \ldots, x_n$ span X, every vector $x \in X$ can be expressed as a linear combination of them; i.e.,

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

for some choice of scalars $\alpha_1, \ldots, \alpha_n$. We now must show that these scalars are unique. To this end, suppose that

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$$

and

$$x = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_n x_n.$$

Then

$$\begin{aligned} x + (-x) &= (\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n) + (-\beta_1 x_1 - \beta_2 x_2 \\ &\quad - \ldots - \beta_n x_n) \\ &= (\alpha_1 - \beta_1)x_1 + (\alpha_2 - \beta_2)x_2 + \ldots + (\alpha_n - \beta_n)x_n = 0. \end{aligned}$$

Since the vectors $x_1, x_2, \ldots, x_n$ form a basis for X, it follows that they are linearly independent, and therefore we must have $(\alpha_i - \beta_i) = 0$ for $i = 1, \ldots, n$. From this it follows that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$. ■

We also have:

3.3.26. Theorem. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for vector space X, and let $\{y_1, \ldots, y_m\}$ be any linearly independent set of vectors. Then $m \leq n$.

Proof. We need to consider only the case $m \geq n$ and prove that then we actually have $m = n$. Consider the set of vectors $\{y_1, x_1, \ldots, x_n\}$. Since the vectors $x_1, \ldots, x_n$ span X, y_1 can be expressed as a linear combination of them. Thus, the set $\{y_1, x_1, \ldots, x_n\}$ is not linearly independent. Therefore, there exist scalars $\beta_1, \alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\beta_1 y_1 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0. \tag{3.3.27}$$

If all the α_i are zero, then $\beta_1 \neq 0$ and $\beta_1 y_1 = 0$. Thus, we can write

$$\beta_1 y_1 + 0 \cdot y_2 + \ldots + 0 \cdot y_m = 0.$$

But this contradicts the hypothesis of the theorem and can't happen because the $y_1, \ldots, y_m$ are linearly independent. Therefore, at least one of the $\alpha_i \neq 0$. Renumbering all the x_i, if necessary, we can assume that $\alpha_n \neq 0$. Solving for x_n we now obtain

$$x_n = \left(\frac{-\beta_1}{\alpha_n}\right)y_1 + \left(\frac{-\alpha_1}{\alpha_n}\right)x_1 + \ldots + \left(\frac{-\alpha_{n-1}}{\alpha_n}\right)x_{n-1}. \tag{3.3.28}$$

Now we show that the set $\{y_1, x_1, \ldots, x_{n-1}\}$ is also a basis for X. Since $\{x_1, \ldots, x_n\}$ is a basis for X, we have $\xi_1, \xi_2, \ldots, \xi_n \in F$ such that

$$x = \xi_1 x_1 + \ldots + \xi_n x_n.$$

Substituting (3.3.28) into the above expression we note that

$$\begin{aligned} x &= \xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_n\left[\left(\frac{-\beta_1}{\alpha_n}\right)y_1 + \ldots + \left(\frac{-\alpha_{n-1}}{\alpha_n}\right)x_{n-1}\right] \\ &= \gamma y_1 + \gamma_1 x_1 + \ldots + \gamma_{n-1}x_{n-1}, \end{aligned}$$

where γ and γ_i are defined in an obvious way. In any case, every $x \in X$ can be expressed as a linear combination of the set of vectors $\{y_1, x_1, \ldots, x_{n-1}\}$, and thus this set must span X. To show that this set is also linearly independent, let us assume that there are scalars $\lambda, \lambda_1, \ldots, \lambda_{n-1}$ such that

$$\lambda y_1 + \lambda_1 x_1 + \ldots + \lambda_{n-1}x_{n-1} = 0,$$

and assume that $\lambda \neq 0$. Then

$$y_1 = \left(\frac{-\lambda_1}{\lambda}\right)x_1 + \ldots + \left(\frac{-\lambda_{n-1}}{\lambda}\right)x_{n-1} + 0 \cdot x_n. \tag{3.3.29}$$

In view of Eq. (3.3.27) we have, since $\beta_1 \neq 0$, the relation

$$y_1 = \left(\frac{-\alpha_1}{\beta_1}\right)x_1 + \ldots + \left(\frac{-\alpha_{n-1}}{\beta_1}\right)x_{n-1} + \left(\frac{-\alpha_n}{\beta_1}\right)x_n. \tag{3.3.30}$$

Now the term $(-\alpha_n/\beta_1)x_n$ in Eq. (3.3.30) is not zero, because we solved for x_n in Eq. (3.3.28); yet the coefficient multiplying x_n in Eq. (3.3.29) is zero. Since $\{x_1, \ldots, x_n\}$ is a basis, we have arrived at a contradiction, in view of Theorem 3.3.25. Therefore, we must have $\lambda = 0$. Thus, we have

$$\lambda_1 x_1 + \ldots + \lambda_{n-1}x_{n-1} + 0 \cdot x_n = 0$$

and since $\{x_1, \ldots, x_n\}$ is a linearly independent set it follows that $\lambda_1 = 0, \ldots, \lambda_{n-1} = 0$. Therefore, the set $\{y_1, x_1, \ldots, x_{n-1}\}$ is indeed a basis for X.

By a similar argument as the preceding one we can show that the set $\{y_2, y_1, x_1, \ldots, x_{n-2}\}$ is a basis for X, that the set $\{y_3, y_2, y_1, x_1, \ldots, x_{n-3}\}$ is a basis for X, etc. Now if $m > n$, then we would not utilize y_{n+1} in our process. Since $\{y_n, \ldots, y_1\}$ is a basis by the preceding argument, there exist coefficients $\eta_n, \ldots, \eta_1$ such that

$$y_{n+1} = \eta_n y_n + \ldots + \eta_1 y_1.$$

But by Theorem 3.3.15 this means the $y_i, i = 1, \ldots, n+1$ are linearly dependent, a contradiction to the hypothesis of our theorem. From this it now follows that if $m \geq n$, then we must have $m = n$. This concludes the proof of the theorem. ■

As a direct consequence of Theorem 3.3.26 we have:

3.3.31. Theorem. If a linear space X has a basis containing a finite number of vectors n, then any other basis for X consists of exactly n elements.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis for X, and let also $\{y_1, \ldots, y_m\}$ be a basis for X. Then in view of Theorem 3.3.26 we have $m \leq n$. Interchanging the role of the x_i and y_i we also have $n \leq m$. Hence, $m = n$. ■

Our preceding result enables us to make the following definition.

3.3.32. Definition. If a linear space X has a basis consisting of a finite number of vectors, say $\{x_1, \ldots, x_n\}$, then X is said to be a **finite-dimensional vector space** and the **dimension of** X is n, abbreviated dim $X = n$. In this case we speak of an **n-dimensional vector space.** If X is not a finite-dimensional vector space, it is said to be an **infinite-dimensional vector space.**

We will agree that the linear space consisting of the null vector is finite dimensional, and we will say that the dimension of this space is zero.

Our next result provides us with an alternate characterization of (finite) dimension of a linear space.

3.3.33. Theorem. Let X be a vector space which contains n linearly independent vectors. If every set of $n + 1$ vectors in X is linearly dependent, then X is finite dimensional and dim $X = n$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a linearly independent set in X, and let $x \in X$. Then there exists a set of scalars $\{\alpha_1, \ldots, \alpha_{n+1}\}$ not all zero, such that

$$\alpha_1 x_1 + \ldots + \alpha_n x_n + \alpha_{n+1} x = 0.$$

Now $\alpha_{n+1} \neq 0$, otherwise we would contradict the fact that $x_1, \ldots, x_n$ are linearly independent. Hence,

$$x = -\left(\frac{\alpha_1}{\alpha_{n+1}}\right)x_1 - \ldots - \left(\frac{\alpha_n}{\alpha_{n+1}}\right)x_n$$

and $x \in V(\{x_1, \ldots, x_n\})$; i.e., $\{x_1, \ldots, x_n\}$ is a basis for X. Therefore, X is n-dimensional. ■

From our preceding result follows:

3.3.34. Corollary. Let X be a vector space. If for given n every set of $n + 1$ vectors in X is linearly dependent, then X is finite dimensional and dim $X \leq n$.

3.3.35. Exercise. Prove Corollary 3.3.34.

We are now in a position to speak of coordinates of a vector. We have:

3.3.36. Definition. Let X be a finite-dimensional vector space, and let $\{x_1, \ldots, x_n\}$ be a basis for X. Let $x \in X$ be represented by

$$x = \xi_1 x_1 + \ldots + \xi_n x_n.$$

The unique scalars $\xi_1, \xi_2, \ldots, \xi_n$ are called the **coordinates of x with respect to the basis** $\{x_1, x_2, \ldots, x_n\}$.

It is possible to prove results similar to Theorems 3.3.26 and 3.3.31 for infinite-dimensional linear spaces. Since we will not make further use of

these results in this book, their proofs will be omitted. In the following theorems, X is an arbitrary vector space (i.e., finite dimensional or infinite dimensional).

3.3.37. Theorem. If Y is a linearly independent set in a linear space X, then there exists a Hamel basis Z for X such that $Y \subset Z$.

3.3.38. Theorem. If Y and Z are Hamel bases for a linear space X, then Y and Z have the same cardinal number.

The notion of Hamel basis is not the only concept of basis with which we will deal. Such other concepts (to be specified later) reduce to Hamel basis on finite-dimensional vector spaces but differ significantly on infinite-dimensional spaces. We will find that on infinite-dimensional spaces the concept of Hamel basis is not very useful. However, in the case of finite-dimensional spaces the concept of Hamel basis is most crucial.

In view of the results presented thus far, the reader can readily prove the following facts.

3.3.39. Theorem. Let X be a finite-dimensional linear space with dim $X = n$.

(i) No linearly independent set in X contains more than n vectors.
(ii) A linearly independent set in X is a basis if and only if it contains exactly n vectors.
(iii) Every spanning or generating set for X contains a basis for X.
(iv) Every set of vectors which spans X contains at least n vectors.
(v) Every linearly independent set of vectors in X is contained in a basis for X.
(vi) If Y is a linear subspace of X, then Y is finite dimensional and dim $Y \leq n$.
(vii) If Y is a linear subspace of X and if dim $X =$ dim Y, then $Y = X$.

3.3.40. Exercise. Prove Theorem 3.3.39.

From Theorem 3.3.39 follows directly our next result.

3.3.41. Theorem. Let X be a finite-dimensional linear space of dimension n, and let Y be a collection of vectors in X. Then any two of the three conditions listed below imply the third condition:

(i) the vectors in Y are linearly independent;
(ii) the vectors in Y span X; and
(iii) the number of vectors in Y is n.

3.3.42. Exercise. Prove Theorem 3.3.41.

Another way of restating Theorem 3.3.41 is as follows:

(a) the dimension of a finite-dimensional linear space X is equal to the smallest number of vectors that can be used to span X; and

(b) the dimension of a finite-dimensional linear space X is the largest number of vectors that can be linearly independent in X.

For the direct sum of two linear subspaces we have the following result.

3.3.43. Theorem. Let X be a finite-dimensional vector space. If there exist linear subspaces Y and Z of X such that $X = Y \oplus Z$, then $\dim(X) = \dim(Y) + \dim(Z)$.

Proof. Since X is finite dimensional it follows from part (vi) of Theorem 3.3.39 that Y and Z are finite-dimensional linear spaces. Thus, there exists a basis, say $\{y_1, \ldots, y_n\}$ for Y, and a basis, say $\{z_1, \ldots, z_m\}$, for Z. Let $W = \{y_1, \ldots, y_n, z_1, \ldots, z_m\}$. We must show that W is a linearly independent set in X and that $V(W) = X$. Now suppose that

$$0 = \sum_{i=1}^{n} \alpha_i y_i + \sum_{i=1}^{m} \beta_i z_i.$$

Since the representation for $0 \in X$ must be unique in terms of its components in Y and Z, we must have

$$\sum_{i=1}^{n} \alpha_i y_i = 0$$

and

$$\sum_{i=1}^{m} \beta_i z_i = 0.$$

But this implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \beta_1 = \beta_2 = \ldots = \beta_m = 0$. Thus, W is a linearly independent set in X. Since X is the direct sum of Y and Z, it is clear that W generates X. Thus, $\dim X = m + n$. This completes the proof of the theorem. ■

We conclude the present section with the following results.

3.3.44. Theorem. Let X be an n-dimensional vector space, and let $\{y_1, \ldots, y_m\}$ be a linearly independent set of vectors in X, where $m < n$. Then it is possible to form a basis for X consisting of n vectors $x_1, \ldots, x_n$, where $x_i = y_i$ for $i = 1, \ldots, m$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for X. Let S_1 be the set of vectors $\{y_1, \ldots, y_m, e_1, \ldots, e_n\}$, where $\{y_1, \ldots, y_m\}$ is a linearly independent set of vectors in X and where $m < n$. We note that S_1 spans X and is linearly

dependent, since it contains more than n vectors. Now let

$$\sum_{i=1}^{m} \alpha_i y_i + \sum_{i=1}^{n} \beta_i e_i = 0.$$

Then there must be some $\beta_j \neq 0$, otherwise the linear independence of $\{y_1, \ldots, y_m\}$ would be contradicted. But this means that e_j is a linear combination of the set of vectors $S_2 = \{y_1, \ldots, y_m, e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n\}$; i.e., S_2 is the set S_1 with e_j eliminated. Clearly, S_2 still spans X. Now either S_2 contains n vectors or else it is a linearly dependent set. If it contains n vectors, then by Theorem 3.3.41 these vectors must be linearly independent in which case S_2 is a basis for X. We then let $x_n = e_j$, and the theorem is proved. On the other hand, if S_2 contains more than n vectors, then we continue the above procedure to eliminate vectors from the remaining e_i's until exactly $n - m$ of them are left. Letting $e_{j_1}, \ldots, e_{j_{n-m}}$ be the remaining vectors and letting $x_{m+1} = e_{j_1}, \ldots, x_n = e_{j_{n-m}}$, we have completed the proof of the theorem. ■

3.3.45. Corollary. Let X be an n-dimensional vector space, and let Y be an m-dimensional subspace of X. Then there exists a subspace Z of X of dimension $(n - m)$ such that $X = Y \oplus Z$.

3.3.46. Exercise. Prove Corollary 3.3.45.

Referring to Figure 3.3.8, it is easy to see that the subspace Z in Corollary 3.3.45 need not be unique.

3.4. LINEAR TRANSFORMATIONS

Among the most important notions which we will encounter are special types of mappings on vector spaces, called **linear transformations.**

3.4.1. Definition. A mapping T of a linear space X into a linear space Y, where X and Y are vector spaces over the same field F, is called a **linear transformation** or **linear operator** provided that

(i) $T(x + y) = T(x) + T(y)$ for all $x, y \in X$; and

(ii) $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and for all $\alpha \in F$.

A transformation which is not linear is called a **non-linear transformation.**

We will find it convenient to write $T \in L(X, Y)$ to indicate that T is a linear transformation from a linear space X into a linear space Y (i.e.,