INTRODUCTIONS TO SYMMETRIC POLYNOMIALS AND SYMMETRIC FUNCTIONS

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1. Introduction

Symmetric polynomials and symmetric functions are ubiquitous in mathematics and mathematical physics. For example, they appear in elementary algebra (e.g. Viete's Theorem), representation theories of symmetric groups and general linear groups over $\mathbb{C}$ or finite fields. They are also important objects to study in algebraic combinatorics.

Via their close relations with representation theory, the theory of symmetric functions has found many applications to mathematical physics. For example, they appear in the Boson-Fermion correspondence which is very important in both superstring theory and the theory of integrable system [2]. They also appear in Chern-Simons theory and the related link invariants and 3-manifold invariants [8]. By the duality between Chern-Simons theory and string theory [9] they emerge again in string theory [1], and in the study of moduli spaces of Riemann surfaces [6].

The following is a revised and expanded version of the informal lecture notes for a undergraduate topic course given in Tsinghua University in the spring semester of 2003. Part of the materials have also been used in a minicourse at the Center of Mathematical Sciences at Zhejiang University as part of the summer program on mathematical physics in 2003. I thank both the audiences for their participation. The purpose of this course is to present an introduction to this fascinating field with minimum prerequisite. I have kept the informal style of the original notes.

2. Symmetric Polynomials

In this section we will give the definition of symmetric polynomials and explain why they are called symmetric.

2.1. **Definitions of symmetric polynomials.**

Let us recall the famous Viete's theorem in elementary algebra. Suppose $x_1, \dots, x_n$ are the n roots of a polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_n.$$

Then

$$\begin{aligned}
&e_1(x_1, \dots, x_n) = \sum_{i=1}^n x_i = -a_1, \\
&e_2(x_1, \dots, x_n) = \sum_{1 \leq i_1 < i_2 \leq n} x_{i_1} x_{i_2} = a_2, \\
&\cdots \\
&e_m(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1} \cdots x_{i_m} = (-1)^m a_m, \\
&\cdots \\
&e_n(x_1, \dots, x_n) = x_1 x_2 \cdots x_n = (-1)^n a_n.
\end{aligned}$$

The polynomial $e_m(x_1, \dots, x_n)$ is called the *m-th symmetric polynomial in* $x_1, \dots, x_n$. It has the following property:

$$e_m(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = e_m(x_1, \dots, x_n),$$

for all permutations σ of $\{1, \dots, n\}$. Recall a permutation of $\{1, \dots, n\}$ is a one-to-one correspondence:

$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}.$$

The above property of $e_m(x_1, \dots, x_n)$ inspires the following:

Definition 2.1. A polynomial $p(x_1, \dots, x_n)$ is called a *symmetric polynomial* if it satisfies:

$$p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = p(x_1, \dots, x_n),$$

for all permutations σ of $\{1, \dots, n\}$. We denote by Λ_n the space of all symmetric polynomials in $x_1, \dots, x_n$.

2.2. **Mathematical description of symmetry.**

Symmetry is clearly a geometric property, so calling a polynomial symmetric might sound strange. To explain the terminology, we need to explain how symmetry is described in mathematics. This involves the algebraic notions of groups and group actions. The concept of a group was introduced by Galois in his study of algebraic solutions of polynomial equations of degree ≥ 5.

2.2.1. *Groups.*

Let us examine a geometric example. A regular pentagon is clearly symmetric, geometrically. The rotation around its center counterclockwise by 72 degrees will take the pentagon to itself. We regard this rotation as a map

$$T : \mathbb{R}^2 \to \mathbb{R}^2,$$

and define for any integer n,

$$T^n = \begin{cases} T \circ T \circ \cdots \circ T(n \text{ times}), & n > 0, \\ \mathrm{id}, & n = 0, \\ T^{-1} \circ T^{-1} \circ \cdots \circ T^{-1}(-n \text{ times}), & n < 0. \end{cases}$$

Note $T^5 = \mathrm{id}$, so we get a set of five elements:

$$G = \{T^n | n \in \mathbb{Z}\}.$$

It is easy to see that the composition of maps defines a map

$$\circ : G \times G \to G.$$

The following properties are clearly satisfied:

$$\begin{aligned} &(T^m \circ T^n) \circ T^r = T^m \circ (T^n \circ T^r), \\ &T^m \circ \mathrm{id} = \mathrm{id} \circ T^m = T^m, \\ &T^m \circ T^{-m} = T^{-m} \circ T^m = \mathrm{id}\,. \end{aligned}$$

Definition 2.2. A *group* is set G together with a map $G \times G \to G$ denoted by $(g_1, g_2) \mapsto g_1 \cdot g_2$, which satisfies the following properties:

(1) $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$, for $g_1, g_2, g_3 \in G$;
(2) there exists an element $e \in G$ (called the identity element), such that

$$e \cdot g = g \cdot e = g$$

for all $g \in G$.
(3) for $g \in G$, there exists an element $g^{-1} \in G$ (called the inverse element of g) such that

$$g \cdot g^{-1} = g^{-1} \cdot g = e.$$

A group is called an *abelian group* if

$$g_1 \cdot g_2 = g_2 \cdot g_1,$$

for $g_1, g_2 \in G$.

Example 2.1. We have seen many examples of groups.

(a) The couples $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$ are abelian groups for which 0 is the identity element.

(b) The couples $(\mathbb{Q}^*, \times)$, $(\mathbb{R}^*, \times)$, and $(\mathbb{C}^*, \times)$ are abelian groups for which 1 is the identity element.

(c) For any integer $n > 1$, the set

$$\mathbb{Z}_n = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\} = \mathbb{Z}/\sim$$

where $p \sim q$ iff $p \equiv q \pmod{n}$, together with $+$, is an abelian group for which $\bar{0}$ is the identity element.

(d) For any integer $n > 1$, the set

$$\mathbb{Z}_n^* = \{\bar{m} \in \mathbb{Z}_n | (m, n) = 1\}$$

together with $\times$ is an abelian group for which $\bar{1}$ is the identity element.

Example 2.2. We also have many examples of groups in linear algebra.

(a) Every vector space is an abelian group under vector additions, with the zero vector as the identity element.

(b) Given a vector space, denote by $GL(V)$ the space of linear transformations from V to itself. Then $GL(V)$ is a group.

(c) Denote by $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$) the space of invertible $n \times n$ matrices with real (or complex) coefficients. Then $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are groups under matrix multiplications.

(d) Denote by $O(n)$ (or $U(n)$) the space of $n \times n$ orthogonal (or unitary) matrices. Then $O(n)$ and $U(n)$ are groups under matrix multiplications.

2.2.2. *Symmetric groups.* A permutation of $[n] = \{1, \ldots, n\}$ can be regarded as a one-to-one correspondence $\sigma : [n] \to [n]$. Denote by S_n the set of all such permutations. Then it is straightforward to see that S_n together the composition of maps $\circ$ is a group, for which the identity element is the identity permutation. It is called the *symmetric group of order* n.

Exercise 2.1. For any set S, denote by $\mathrm{Aut}(S)$ the set of one-to-one correspondences $\rho : S \to S$. Prove $(\mathrm{Aut}(S), \circ)$ is a group for which the identity map is the identity element. Here $\circ$ denotes the composition of maps.

2.2.3. *Group actions.* A closely related concept is that of a group action. A motivating example is that an element in S_n permutes the n roots of a polynomial of degree n.

Definition 2.3. Let G be a group. A G-action on a set S is a map $G \times S \to S$ denoted by $(g, s) \in G \times S \mapsto g \cdot s \in S$, which satisfies the following properties:

(1) $g_1 \cdot (g_2 \cdot s) = (g_1 \cdot g_2) \cdot s$, for $g_1, g_2 \in G$, $s \in S$;
(2) $e \cdot s = s$, for $s \in S$.

Given a group action $G \times S \to S$, for any $g \in G$, the assignment $s \mapsto g \cdot s$ defines a map $\rho(g)$. It is easy to see that $\rho(g) \in \mathrm{Aut}(S)$ for $\rho(g^{-1})$ is the inverse of $\rho(g)$. The assignment $g \mapsto \rho(g)$ defines a map $\rho : G \to \mathrm{Aut}(S)$. It satisfies:

$$\rho(g_1 \cdot g_2) = \rho(g_1) \circ \rho(g_2),$$

for $g_1, g_2 \in G$. This inspires the following:

Definition 2.4. A *group homomorphism* between two two groups G_1 and G_2 is a map $\rho : G_1 \to G_2$ satisfying the following property:

$$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2)$$

for all $g_1, g_2 \in G_1$.

Conversely, given a map $\rho : G \to \mathrm{Aut}(S)$ satisfying the above properties, one gets a G-action on S by defining

$$g \cdot s = \rho(g)(s),$$

for $g \in G$, $s \in S$. Hence in the following we will use a map $\rho : G \to \mathrm{Aut}(S)$ to denote a group action.

2.2.4. *Fixed point set.*

Definition 2.5. Let $\rho : G \to \mathrm{Aut}(S)$ be a group action. An element $s \in G$ is called a *fixed point* if

$$g \cdot s = s,$$

for all $g \in G$. We denote the set of fixed points in S by S^G.

Exercise 2.2. Let V be a vector space. A homomorphism $\rho : G \to GL(V)$ is called a *representation* of G on V. Given a representation of G on V, V^G is a linear subspace of V.

2.3. **Symmetric polynomials as fixed points.** We now return to the discussion of symmetric polynomials. Denote by $\mathbb{C}[x_1, \dots, x_n]$ the set of polynomials in $x_1, \dots, x_n$ of complex coefficients. Define a map

$$S_n \times \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$$

by

$$(\sigma \cdot p)(x_1, \dots, x_n) = p(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}),$$

for $\sigma \in S_n$, $p \in \mathbb{C}[x_1, \dots, x_n]$. We leave the proof of the following Proposition as an exercise:

Proposition 2.1. *The above map defines a representation of S_n on $\mathbb{C}[x_1, \dots, x_n]$. The fixed points of this action are exactly the set of symmetric polynomials in $x_1, \dots, x_n$.*

3. Poincaré series of Λ_n

We will show Λ_n has a natural structure of a graded vector space for which we can define its Poincaré series.

3.1. A natural grading of $\mathbb{C}[z_1, \dots, x_n]$. For simplicity of notations, we will write

$$R_n = \mathbb{C}[x_1, \dots, x_n].$$

Recall the *degree* of a monomial $x_1^{i_1} \cdots x_n^{i_n}$ is

$$i_1 + \cdots + i_n.$$

A polynomial

$$p(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

is said to be homogeneous of degree k if $a_{i_1, \dots, i_n} \neq 0$ only when $i_1 + \cdots + i_n = k$. Denote by R_n^k the space of all homogeneous polynomials in $x_1, \dots, x_n$ of degree k. The following characterization of homogeneous polynomials are well-known:

Proposition 3.1. *The following statements are equivalent for $p \in R_n$:*
(1) $p \in \Lambda_n^k$.
(2) $p(\lambda x_1, \dots, \lambda x_n) = \lambda^k p(x_1, \dots, x_n)$, for all $\lambda \in \mathbb{C}$.
(3) $x_1 \frac{\partial p}{\partial x_1} + \cdots + x_n \frac{\partial p}{\partial x_n} = kp$.

Remark 3.1. The vector field $E = x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$ is called the Euler vector field on $\mathbb{C}^n$.

The fact that every polynomial can be uniquely written as a sum of homogeneous polynomials implies:

$$R_n = \oplus_{k \geq 0} R_n^k.$$

Lemma 3.1. *One has*

$$\dim R_n^k = \binom{k+n-1}{n-1}.$$

Proof. Clearly

$$\{x_1^{i_1} \cdots x_n^{i_n} | i_1 + \cdots + i_n = k, i_1, \dots, i_n \geq 0\}$$

is a basis of R_n^k, hence we need to find the number of sequences $(i_1, \dots, i_n)$ of nonnegative integers with $i_1 + \cdots + i_n = k$. Each of such a sequence corresponds to a picture of the following form

$$\underbrace{\circ \cdots \circ}_{i_1} \bullet \underbrace{\circ \cdots \circ}_{i_2} \bullet \cdots \bullet \underbrace{\circ \cdots \circ}_{i_n}$$

I.e., among a sequence of $k + n - 1$ whites balls, change $n - 1$ balls by black balls. The number of ways doing this is exactly given by $\binom{k+n-1}{n-1}$. $\square$

Corollary 3.1. *The sequence $\dim R_n^k$ has the following generating function:*

$$\sum_{k \geq 0} \dim R_n^k \, t^k = \frac{1}{(1-t)^n}. \tag{1}$$

Proof. Expand the right-hand side of (1) as Taylor series:

$$\frac{1}{(1-t)^n} = \sum_{k\geq 0} \binom{k+n-1}{k} t^k = \sum_{k\geq 0} \binom{k+n-1}{n-1} t^k = \sum_{k\geq 0} \dim R_n^k \, t^k.$$

□

3.2. **Graded vector spaces and Poincaré series.** The above discussions for R_n inspires the following:

Definition 3.1. A ($\mathbb{Z}$-)*grading* on a vector space V is a direct sum decomposition

$$V = \oplus_{k\in\mathbb{Z}} V^k.$$

If $v \in V^k$, then we write $\deg v = k$. A *graded vector space* is a vector space with a grading. Suppose V is a graded vector space with $V^k = 0$ for $k < 0$ and $\dim V^k < \infty$ for $k \geq 0$, we define its *Poincaré series* by

$$p_t(V) = \sum_{k\geq 0} \dim V^k \, t^k$$

Exercise 3.1. Let V and W be two graded vector space. Define the following grading on $V \oplus W$ and $V \otimes W$:

$$(V \oplus W)^k = V^k \oplus W^k, \tag{2}$$

$$(V \otimes W)^k = \oplus_{p+q=k} V^p \otimes W^q. \tag{3}$$

Suppose $p_t(V)$ and $p_t(W)$ can be defined, then $p_t(V \oplus W)$ and $p_t(V \times W)$ can defined and we have

$$p_t(V \oplus W) = p_t(V) + p_t(W),$$
$$p_t(V \otimes W) = p_t(V) p_t(W).$$

3.3. **A natural grading on Λ_n.** For a nonnegative integer k, denote by Λ_n^k the space of homogeneous symmetric polynomials in $x_1, \ldots, x_n$ of degree k. Clearly

$$\Lambda_n^k = \Lambda_n \cap R_n^k,$$

and

$$\Lambda_n = \oplus_{k\geq 0} \Lambda_n^k.$$

We are interested in computing $\dim \Lambda_n^k$ and their generating function. For this purpose we have to first find a basis of Λ_n^k. As will be seen later, there are many natural choices of bases, and the study of relationships among them is an important aspect of the theory of symmetric functions. For now, we construct a basis from the monomial as follows. Define a map $S : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ by

$$(Sp)(x_1, \ldots, x_n) = \sum_{\sigma\in S_n} p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Then clearly $Sp \in \Lambda_n$. In particular, for a sequence of nonnegative integers $(\lambda_1, \ldots, \lambda_n)$,

$$S(x_1^{\lambda_1} \cdots x_n^{\lambda_n}) = \sum_{\sigma\in S_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n}$$

is a symmetric polynomial. Since we also have

$$S(x_1^{\lambda_1} \cdots x_n^{\lambda_n}) = \sum_{\sigma\in S_n} x_1^{\lambda_{\sigma(1)}} \cdots x_n^{\lambda_{\sigma(n)}},$$

it follows that for any permutation $(\tilde{\lambda}_1, \dots, \tilde{\lambda}_n)$ of $(\lambda_1, \dots, \lambda_n)$, we have

$$S(x_1^{\tilde{\lambda}_1} \cdots x_n^{\tilde{\lambda}_n}) = S(x_1^{\lambda_1} \cdots x_n^{\lambda_n}).$$

3.4. **Partitions.** Hence we will only consider sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

We will write

$$|\lambda| = \lambda_1 + \cdots + \lambda_n.$$

When $|\lambda| = k$, we will say such a λ is a *partition* of k, and write $\lambda \vdash k$. The number of nonzero λ_i's is called the *length* of λ, and is denoted by $l(\lambda)$. We will use the set

$$\mathcal{P}(k, n) = \{\lambda \vdash k | l(\lambda) \leq n\}$$

of partitions of k with length $\leq n$. For a partition λ, define

$$m_i(\lambda) = |\{j | \lambda_j = i\}|.$$

We will often write a partition also in the form of

$$1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots k^{m_k(\lambda)} \cdots$$

In this notation, we then have

$$\mathcal{P}(k, n) = \{1^{m_1} \cdots k^{m_k} | \sum_{i=1}^{k} i m_i = k, \sum_i m_i \leq n\}.$$

In the following, we will often omit the zero's in a partition.

Example 3.1. (1) There is only one partition of 1: $1 = 1$, it has length $l(\lambda) = 1$, and $m_1(\lambda) = 1$, $m_i(\lambda) = 0$ for $i \geq 2$, hence it can also be written as 1^1.

(2) There are two partitions of 2: (2) and $(1, 1)$. For $\lambda = (2)$, we have $l(\lambda) = 1$, $m_2(\lambda) = 1$, $m_i(\lambda) = 0$ for $i \neq 2$, hence λ can also be written as 2^1. For $\lambda = (1, 1)$, we have $l(\lambda) = 2$, $m_1(\lambda) = 2$, $m_i(\lambda) = 0$ for $i > 1$, hence λ can also be written as 1^2.

(3) There are three partitions of 3. For $\lambda = (3)$, $l(\lambda) = 1$, $m_3(\lambda) = 1$, $m_i(\lambda) = 0$ for $i \neq 3$, hence λ can also be written as 3^1. For $\lambda = (2, 1)$, we have $l(\lambda) = 2$, $m_1(\lambda) = m_2(\lambda) = 1$, $m_i(\lambda) = 0$ for $i > 2$, hence λ can also be written as $1^1 2^1$. For $\lambda = (1, 1, 1)$, we have $l(\lambda) = 3$, $m_1(\lambda) = 3$, $m_i(\lambda) = 0$ for $i > 1$, hence λ can also be written as 1^3.

3.5. **Monomial symmetric polynomials.** For a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers, define

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}(k, n)$, consider the symmetric polynomial $S(x^\lambda)$. For example, let $\lambda = (1, 1)$ (where we have omitted $n - 2$ zero's), then in $\mathbb{C}[x_1, \dots, x_n]$ we have

$$S(x_1 x_2) = 2(n-2)! \sum_{1 \leq i < j \leq n} x_i x_j.$$

It is more natural to ignore the factor $2(n-2)!$ and consider

$$\sum_{1 \leq i < j \leq j \leq n} x_i x_j.$$

In general for $\lambda \in \mathcal{P}(k, n)$, define

$$m_\lambda(x_1, \dots, x_n) = \sum x^\alpha,$$

where the sum is taken over all *distinct* permutations of λ. This is a symmetric polynomial, called the *monomial symmetric polynomial* corresponding to λ.

Example 3.2. For example,

$$\begin{aligned} m_{(k)}(x_1,\dots,x_n) &= x_1^k+\cdots+x_n^k, \\ m_{(3,2)}(x_1,\dots,x_n) &= \sum_{1\le i<j\le n}(x_i^3x_j^2+x_i^3x_j^2), \\ m_{(2,2)}(x_1,\dots,x_n) &= \sum_{1\le i<j\le n} x_i^2x_j^2. \end{aligned}$$

We leave the proof of the following Proposition as an exercise:

Proposition 3.2. *The set*

$$\{m_\lambda(x_1,\dots,x_n)|\lambda\in\mathcal{P}(k,n)\}$$

is a basis of Λ_n^k, *hence*

$$\dim \Lambda_n^k = |\mathcal{P}(k,n)|.$$

3.6. **Young diagrams.** In the above we have reduced the problem of finding the Poincaré series of Λ_n to the problem of finding the generating series of the numbers of partitions of length $\le n$. The latter is still not easy to solve at first sight. But there is a related problem that has a very easy solution, i.e., the problem of finding the numbers of partitions whose parts are $\le n$, more precisely, the problem of find the number of elements in the following set:

$$\mathcal{P}'(k,n) = \{1^{m_1}\cdots n^{m_n}|\sum_{i=1}^n im_i = k\}.$$

We have

Proposition 3.3. *The generating function of* $|\mathcal{P}'(k,n)|$ *is*

$$\sum_{k=0}^\infty |\mathcal{P}'(k,n)|t^k = \frac{1}{\prod_{i=1}^n(1-t^i)}.$$

Proof. This is proved by using the series expansion:

$$\frac{1}{1-t} = \sum_{m=0}^\infty t^m$$

as follows.

$$\frac{1}{\prod_{i=1}^n(1-t^i)} = \prod_{i=1}^n\sum_{m_i=0}^\infty t^{im_i} = \sum_{m_1,\dots,m_n=0}^\infty t^{\sum_{i=1}^n im_i} = \sum_{k=0}^\infty |\mathcal{P}'(k,n)|t^k.$$

□

We will find a one-to-one correspondence between $\mathcal{P}(k,n)$ and $\mathcal{P}'(k,n)$. This can be achieved by exploiting a graphical representation of a partition as follows. Given a partition λ, the *Young diagram of* λ consists of $l(\lambda)$ rows of adjacent squares: the i-th row has λ_i squares, $i=1,\dots,l(\lambda)$. The first square of each row lies at the same column. We will often denote also by λ the Young diagram of λ. It is clear that λ has $|\lambda|$ squares. The *transpose* of a Young diagram λ, denoted by λ', is the Young diagram obtained by transposing the columns and rows of λ.

Now $\mathcal{P}(k,n)$ corresponds to the set of Young diagrams with k squares and $\leq n$ rows, $\mathcal{P}'(k,n)$ corresponds to the set of Young diagrams with k squares and $\leq n$ columns. Hence the map $\lambda \mapsto \lambda'$ establishes a one-to-one correspondence between $\mathcal{P}(k,n)$ and $\mathcal{P}'(k,n)$.

Corollary 3.2. *The Poincaré series of the Λ_n has the following generating function:*

$$p_t(\Lambda_n) = \frac{1}{\prod_{i=1}^n (1-t^i)}. \tag{4}$$

4. Ring generators of Λ_n

In this section we will give interpretations of formulas (1) and (4) in terms of rings and generators.

4.1. **Rings.** On the space Λ_n of symmetric polynomials in $x_1, \ldots, x_n$, one can define not only the addition, but also the multiplication. The standard properties of additions and multiplications of numbers are satisfied.

Definition 4.1. A ring is a set R together with two maps

$$+ : R \times R \to R$$

and

$$\cdot : R \times R \to R,$$

with the following properties: $(R, +)$ is an abelian group, and

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),$$
$$(x + y) \cdot z = x \cdot z + y \cdot z,$$
$$z \cdot (x + y) = z \cdot x + z \cdot y,$$

for $x, y, z \in R$. A ring R is said to be *commutative* if

$$x \cdot y = y \cdot x,$$

for $x, y \in R$. An *identity of a ring* is an element $1 \in R$ such that

$$1 \cdot x = x \cdot 1 = x,$$

for all $x \in R$. A *graded ring* is a ring R with a decomposition

$$R = \oplus R_n$$

such that each R_n is closed under $+$, and

$$R_m \cdot R_n \subset R_{m+n}.$$

Example 4.1. (1) $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, and $(\mathbb{C}, +, \cdot)$ are all commutative rings with identity.

(2) For any positive integer n, $(\mathbb{Z}_n, +, \cdot)$ is a commutative ring with identity.

(3) For any positive integer n, the space $\mathbb{C}[x_1, \ldots, x_n]$ of polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$ is a graded commutative ring with identity under additions and multiplications of polynomials.

For us the following Proposition is relevant:

Proposition 4.1. *For any positive integer n, $(\Lambda_n, +, \cdot)$ is a graded commutative ring with identity.*

4.2. **Ring generators and Poincaré series.**

Definition 4.2. A ring R is said to be *freely generated* by elements $a_1, \ldots, a_n \in R$ over $\mathbb{C}$ if every element can be uniquely written as a polynomials in $a_1, \ldots, a_n$. In this case, we will write

$$R = \mathbb{C}[a_1, \ldots, a_n].$$

Theorem 4.1. *Suppose R is a graded ring freely generated by homogeneous elements $a_1, \ldots, a_n$, and $\deg a_i = m_i$, $i = 1, \ldots, n$. Then we have*

$$p_t(R) = \frac{1}{\prod_{i=1}^n (1 - t^{m_i})}.$$

Proof. Note
$$\mathbb{C}[a_i] = \oplus \mathbb{C} a_i^k,$$
hence it is straightforward to see that
$$p_t(\mathbb{C}[a_i]) = \sum_{k=0} t^{km_i} = \frac{1}{1 - t^{m_i}}.$$
It is easy to see that as graded vector spaces,
$$R = \mathbb{C}[a_1] \otimes \cdots \mathbb{C}[a_n].$$
Hence by (3),
$$p_t(R) = \prod_{i=1}^n p_t(\mathbb{C}[a_i]) = \frac{1}{\prod_{i=1}^n (1 - t^{m_i})}.$$
□

For example, $\mathbb{C}[x_1, \dots, x_n]$ is freely generated by $x_1, \dots, x_n$ over $\mathbb{C}$, and all x_i have degree 1. Hence
$$p_t(\mathbb{C}[x_1, \dots, x_n]) = \frac{1}{(1-x)^n}.$$
This recovers (1).

4.3. **Application to Λ_n.** In view of Theorem 4.1, (4) is a corollary of the following:

Theorem 4.2. *For any positive integer n,*
$$\Lambda_n = \mathbb{C}[e_1, \dots, e_n].$$

Before we give a proof, note this Theorem means that
$$\{e_1^{m_1} \cdots e_m^{m_n} | m_i \in \mathbb{Z}_+\}$$
is a basis of Λ_n. Here $\mathbb{Z}_+$ is the set of nonnegative integers. Now $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ corresponds to a partition $\lambda' \in \mathcal{P}'(k, n)$, where
$$k = \sum_{i=1}^n i m_i.$$
For $\lambda' \in \mathcal{P}'(k, n)$, define
$$e_{\lambda'}(x_1, \dots, x_n) = e_1^{m_1}(x_1, \dots, x_n) \cdots e_n^{m_n}(x_1, \dots, x_n).$$
If $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_k)$, then clearly we have
$$e_{\lambda'} = e_{\lambda'_1} \cdots e_{\lambda'_k}.$$
Hence to prove Theorem 4.2, it suffices to show that
$$\{e_{\lambda'}\}_{\lambda' \in \mathcal{P}'(k,n)}$$
is a basis of Λ_n^k. Recall
$$\{m_\lambda\}_{\lambda \in \mathcal{P}(k,n)}$$
is a basis of Λ_n^k. Hence Theorem 4.2 follows from the following:

Lemma 4.1. *Let $\lambda \in \mathcal{P}(k,n)$, then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu} a_{\lambda\mu} m_\mu,$$

for some nonnegative integers $a_{\lambda\mu}$, where the sum is over over partition $\mu < \lambda$ (i.e., the first nonvanishing $\lambda_i - \mu_i$ is positive).

This can be proved by using the lexicographic order of polynomials. It will be left to the interested reader.

5. Complete Symmetric Polynomials

5.1. **Complete symmetric polynomials and their generating function.** For each $k \geq 0$, the *complete symmetric polynomial* is the sum of all monomials of degree k:

$$h_k(x_1, \dots, x_n) = \sum_{d_1+\cdots+d_n=k} x_1^{d_1} \cdots x_n^{d_n}.$$

In particular $h_0(x_1, \dots, x_n) = 1$. It is not hard to see that

$$h_k(x_1, \dots, x_n) = \sum_{\lambda \in \mathcal{P}(k,n)} m_\lambda(x_1, \dots, x_n).$$

Define the generating function for h_k by:

$$H_n(t) = \sum_{k \geq 0} h_k(x_1, \dots, x_n) t^k$$

Then we have

$$H_n(t) = \sum_{d_1,\dots,d_n \geq 0} x_1^{d_1} \cdots x_n^{d_n} t^{d_1+\cdots+d_n} = \frac{1}{\prod_{i=1}^n (1 - tx_i)}. \tag{5}$$

5.2. **Relationship with elementary symmetric polynomials.** Consider the generating function of elementary symmetric polynomials:

$$E_n(t) = \sum_{i=0}^n e_i(x_1, \dots, x_n) t^i = \prod_{i=1}^n (1 + tx_i). \tag{6}$$

Celarly we have

$$H(t)E(-t) = 1,$$

or equivalently,

$$\sum_{r=0}^k (-1)^r e_r h_{n-r} = 0 \tag{7}$$

for all $k \geq 1$. Here we have set

$$e_r(x_1, \dots, x_n) = 0$$

for $r > n$.

5.3. **Determinantal formulas.** We now solve (7) inductively. For $k = 1$,

$$h_1 - e_1 = 0$$

hence

$$h_1 = e_1.$$

For $k = 2$,

$$h_2 - e_1 h_1 + e_2 = 0,$$

hence

$$h_2 = e_1^2 - e_2 = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix}$$

Inductively one finds:

$$h_k = \begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_{k-1} & e_k \\ 1 & e_1 & e_2 & \cdots & e_{k-2} & e_{k-1} \\ 0 & 1 & e_1 & & e_{k-3} & e_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{vmatrix} = \det(e_{1-i+j})_{1\le i,j\le n}. \tag{8}$$

By symmetry between h and e in the formula (7), one also get

$$e_k = \begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_{k-1} & h_k \\ 1 & h_1 & h_2 & \cdots & h_{k-2} & h_{k-1} \\ 0 & 1 & h_1 & & h_{k-3} & h_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & h_1 & h_2 \\ 0 & 0 & 0 & \cdots & 1 & h_1 \end{vmatrix} = \det(h_{1-i+j})_{1\le i,j\le n}. \tag{9}$$

Here we have used the convention that

$$e_i(x_1, \ldots, x_n) = 0$$

for $i < 0$ or $i > n$.

5.4. **An involution on Λ_n.** The symmetry between h and e suggests the introduction of the following map $\omega : \Lambda_n \to \Lambda_n$:

$$\omega(\sum_{m_1,\ldots,m_n} a_{m_1,\ldots,m_n} e_1^{m_1} \cdots e_n^{m_n}) = \sum_{m_1,\ldots,m_n} a_{m_1,\ldots,m_n} h_1^{a_1} \cdots h_n^{m_n}.$$

It has the following properties:

(a) ω is a ring homomorphism, i.e.

$$\omega(p+q) = \omega(p) + \omega(q), \qquad \omega(p \cdot q) = \omega(p) \cdot \omega(q),$$

for $p, q \in \Lambda_n$.

(b) $\omega(e_i) = h_i$ and $\omega(h_i) = e_i$.

(c) $\omega^2 = \mathrm{id}$.

(a) is trivial. The first identity in (b) is by definition. For the second identity, apply ω on both sides of (8) then use (9). (c) is a straightforward consequence of (a) and (b).

As a corollary, we see that

$$\Lambda_n = \mathbb{C}[h_1, \ldots, h_n].$$

In other words, if we define for $\lambda' = (\lambda'_1, \ldots, \lambda'_k) \in \mathcal{P}'(k, n)$,

$$h_{\lambda'}(x_1, \ldots, x_n) = h_{\lambda'_1}(x_1, \ldots, x_n) \cdots h_{\lambda'_k}(x_1, \ldots, x_n),$$

then

$$\{h_{\lambda'}\}_{\lambda' \in \mathcal{P}'(k,n)}$$

is a basis of Λ_n^k.

6. Newton Polynomials

6.1. **Newton polynomials and their generating function.** For $r \geq 1$, the r-th *Newton polynomial (power sum)* in $x_1, \dots, x_n$ is

$$p_r(x_1, \dots, x_n) = x_1^r + \cdots + x_n^r.$$

The generating function for them is

$$\begin{aligned} P_n(t) &= \sum_{r \geq 1} p_r(x_1, \dots, x_n) t^{r-1} = \sum_{i=1}^n \sum_{r \geq 1} x_i^r t^{r-1} \\ &= \sum_{i=1}^n \frac{x_i}{1 - x_i t} = \frac{d}{dt} \log \frac{1}{\prod_{i=1}^n (1 - x_i t)}. \end{aligned} \tag{10}$$

6.2. **Newton formulas.** By comparing with (5) and (6), one gets:

$$P_n(t) = \frac{H_n'(t)}{H_n(t)} = \frac{E_n'(-t)}{E_n(-t)}.$$

By applying ω, one gets:

$$\omega(P_n(t)) = P_n(-t),$$

or equivalently,

$$\omega(p_r) = (-1)^{r-1} p_r.$$

One also has

$$H_n'(t) = P_n(t) H_n(t), \qquad E_n'(t) = P_n(-t) E_n(t).$$

Equivalently,

$$k h_k = \sum_{r=1}^k p_r h_{k-r}, \tag{11}$$

$$k e_k = \sum_{r=1}^k (-1)^{r-1} p_r e_{k-r}. \tag{12}$$

These are called the *Newton formulas*.

6.3. **Determinantal formulas.** We now inductively solve (12). For $k = 1$,

$$e_1 = p_1.$$

For $k = 2$,

$$2e_2 = p_1 e_1 - p_2 = \begin{vmatrix} p_1 & p_2 \\ 1 & p_1 \end{vmatrix}.$$

For $k = 3$,

$$3! e_3 = 2 p_1 e_2 - p_1 e_1 + p_2 = \begin{vmatrix} p_1 & p_2 & p_3 \\ 2 & p_1 & p_2 \\ 0 & 1 & p_1 \end{vmatrix}.$$

By induction, one finds

$$k!e_k = \begin{vmatrix} p_1 & p_2 & p_3 & \cdots & p_{k-1} & p_k \\ k-1 & p_1 & p_2 & \cdots & p_{k-2} & p_{k-1} \\ 0 & k-2 & p_1 & \cdots & p_{k-3} & p_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_1 & p_2 \\ 0 & 0 & 0 & \cdots & 1 & p_1 \end{vmatrix} \tag{13}$$

One can also rewrite (12) as

$$p_k = \sum_{r=1}^{k-1} (-1)^{k-r-1} e_{k-r} p_r + (-1)^{k-1} k e_k.$$

For $k = 1$,

$$p_1 = e_1,$$

For $k = 2$,

$$p_2 = e_1 p_1 - 2e_2 = \begin{vmatrix} e_1 & 2e_2 \\ 1 & e_1 \end{vmatrix}.$$

For $k = 3$,

$$p_3 = e_1 p_2 - e_2 p_1 + 3e_3 = \begin{vmatrix} e_1 & e_2 & 3e_3 \\ 1 & e_1 & 2e_2 \\ 0 & 1 & e_1 \end{vmatrix}.$$

By induction, one finds

$$p_k = \begin{vmatrix} e_1 & e_2 & e_3 & \cdots & e_{k-1} & ke_k \\ 1 & e_1 & e_2 & \cdots & e_{k-2} & (k-1)e_{k-1} \\ 0 & 1 & e_1 & \cdots & e_{k-3} & (k-2)e_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 2e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{vmatrix} \tag{14}$$

By applying ω on both sides of (13) and (14), one gets:

$$k!h_k = \begin{vmatrix} p_1 & p_2 & p_3 & \cdots & p_{k-1} & p_k \\ -(k-1) & p_1 & p_2 & \cdots & p_{k-2} & p_{k-1} \\ 0 & -(k-2) & p_1 & \cdots & p_{k-3} & p_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_1 & p_2 \\ 0 & 0 & 0 & \cdots & -1 & p_1 \end{vmatrix} \tag{15}$$

and

$$(-1)^{k-1}p_k = \begin{vmatrix} h_1 & h_2 & h_3 & \cdots & h_{k-1} & kh_k \\ 1 & h_1 & h_2 & \cdots & h_{k-2} & (k-1)h_{k-1} \\ 0 & 1 & h_1 & \cdots & h_{k-3} & (k-2)h_{k-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & h_1 & 2h_2 \\ 0 & 0 & 0 & \cdots & 1 & h_1 \end{vmatrix}. \tag{16}$$

As a corollary, we have

$$\Lambda_n = \mathbb{C}[p_1, \ldots, p_n].$$

(This is a straightforward consequence of (13) and (14).) In other words, if we define for $\lambda' = (\lambda'_1, \ldots, \lambda'_k) \in \mathcal{P}'(k, n)$,

$$p_{\lambda'}(x_1, \ldots, x_n) = p_{\lambda'_1}(x_1, \ldots, x_n) \cdots p_{\lambda'_k}(x_1, \ldots, x_n),$$

then

$$\{p_{\lambda'}\}_{\lambda' \in \mathcal{P}'(k,n)}$$

is a basis of Λ_n^k.

7. Schur Polynomials

So far we have only considered symmetric polynomials. In this section we will consider anti-symmetric polynomials, and their relations with symmetric polynomials. This leads us to the Schur polynomials.

7.1. **Anti-symmetric polynomials.** Recall a permutation can be written as a product of transpositions. Consider the parity of the number of transpositions in such a product. If it is even (or odd), then we say the permutation is even (or odd). The sign of a permutation is defined by:

$$(-1)^\sigma = \begin{cases} 1, & \sigma \text{ is even,} \\ -1, & \sigma \text{ is odd.} \end{cases}$$

Definition 7.1. A polynomial $p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be antisymmetric if

$$p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^\sigma p(x_1, \ldots, x_n),$$

for $\sigma \in S_n$. Denote by A_n the space of all antisymmetric polynomials in $x_1, \ldots, x_n$.

Example 7.1. For any nonnegative integers $d_1, \ldots, d_n$, let

$$d = (d_1, \ldots, d_n)$$

and

$$a_d(x_1, \ldots, x_n) = \begin{vmatrix} x_1^{d_1} & x_1^{d_2} & \cdots & x_1^{d_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ x_n^{d_1} & x_n^{d_2} & \ldots & x_n^{d_n} \end{vmatrix}$$

is antisymmetric. In particular, let $\delta = (n-1, n-2, \ldots, 1, 0)$, the Vandermonde determinant

$$\Delta(x_1, \ldots, x_n) = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_n^{n-1} & x_n^{n-2} & \ldots & x_n & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j)$$

is antisymmetric.

It is straightforward to verify the following:

Proposition 7.1. *As a subspace of* $\mathbb{C}[x_1, \ldots, x_n]$*, the space* A_n *has the following properties:*

- A_n *is closed under additions.*
- $A_n \cdot A_n \subset \Lambda_n$ *and* $\Lambda_n \cdot A_n \subset A_n$.

It is instructive to recall the following definitions from algebra.

Definition 7.2. Let R be a commutative ring, an R-module is an abelian group $(M, +)$ together with a map $R \times M \to M$ denoted by

$$(a, x) \mapsto a \cdot x$$

for $a \in R$, $x \in M$, such that

$$a \cdot (x_1 + x_2) = a \cdot x_1 + a \cdot x_2,$$
$$(a_1 + a_2) \cdot x = a_1 \cdot x + a_2 \cdot x,$$
$$(a_1 \cdot a_2) \cdot x = a_1 \cdot (a_2 \cdot x),$$

for $a, a_1, a_2 \in R$, $x, x_1, x_2 \in M$.

One can easily define R-module homomorphisms and isomorphisms.

Example 7.2. Let R be a commutative ring, then $R^{\oplus n} = R \oplus \cdots R$ (n times) is automatically an R-module. An R-module isomorphic to $R^{\oplus n}$ is called a free R-module of rank n.

Definition 7.3. A superalgebra over $\mathbb{C}$ is a $\mathbb{C}$ algebra A with a $\mathbb{Z}_2$-grading

$$A = A^0 \oplus A^1$$

such that

$$A^0 \cdot A^0 \subset A^0, \qquad A^0 \cdot A^1 \subset A^1, \qquad A^1 \cdot A^0 \subset A^1, \qquad A^1 \cdot A^1 \subset A^0.$$

A superalgebra is said to supercommutative if

$$a \cdot b = (-1)^{|a| \cdot |b|} b \cdot a,$$

for homogeneous element in A. Recall an element $a \in A$ is said to be homogeneous if $a \in A^i$, and in this case we will write $|a| = i$.

Hence A_n is a Λ_n-module. Furthermore, $\Lambda \oplus A_n$ is a superalgebra, though not supercommutative.

7.2. **A_n as a Λ_n-module.** The main result of this subsection is the following:

Theorem 7.1. *For any positive integer n, A_n is a free Λ_n-module of rank 1.*

We will need the following easy Lemma:

Lemma 7.1. *Suppose $p(x) \in \mathbb{C}[x]$. Then*

$$p(a) = 0$$

for some $a \in \mathbb{C}$ if and only if $(x - a) | p(x)$.

Theorem 7.2. *Given $p(x_1, \dots, x_n) \in \Lambda_n$, we have*

$$\Delta(x_1, \dots, x_n) p(x_1, \dots, x_n) \in A_n.$$

Conversely, given any $q(x_1, \dots, x_n) \in A_n$,

$$\frac{q(x_1, \dots, x_n)}{\Delta(x_1, \dots, x_n)} \in \Lambda_n.$$

Proof. The first statement is obvious. For the second statement, we first show that

$$q(x_1, \dots, x_n)|_{x_i = x_j} = 0, \tag{17}$$

for $1 \leq i < j \leq n$. Indeed, let σ be the transposition of i and j. Then for $q \in A_n$ we have

$$q(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = -q(x_1, \dots, x_j, \dots, x_i, \dots, x_n).$$

One easily gets (17) by taking $x_i = x_j$. Hence by Lemma 7.1,

$$(x_i - x_j) | q(x_1, \dots, x_n),$$

therefore,

$$\Delta(x_1,\dots,x_n)|q(x_1,\dots,x_n).$$

It is straightforward to see that q/Δ is symmetric. □

Now we construct maps $F:\Lambda_n\to A_n$ and $G:A_n\to\Lambda_n$ by:

$$F(p)=p\Delta, \qquad G(q)=q/\Delta,$$

for $p\in\Lambda_n$, $q\in A_n$. It is easy to see that F and G are Λ_n-module homomorphisms and they are inverse to each other. This proves Theorem 7.1.

7.3. **Schur polynomials.** Similar to the introduction of monomial symmetric polynomials, one can introduce the "antisymmetric monomials" as follows. Introduce an operator

$$A:\mathbb{C}[x_1,\dots,x_n]\to\mathbb{C}[x_1,\dots,x_n]$$

by

$$(Ap)(x_1,\dots,x_n)=\sum_{\sigma\in S_n}(-1)^\sigma p(x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

It is easy to see that $Ap\in A_n$ for all $p\in\mathbb{C}[x_1,\dots,x_n]$. For example,

$$A(x_1^{d_1}\cdots x_n^{d_n})=\sum_{\sigma\in S_n}(-1)^\sigma x_{\sigma(1)}^{d_1}\cdots x_{\sigma_n}^{d_n}=\Delta_d(x_1,\dots,x_n),$$

where $d=(d_1,\dots,d_n)$.

Now similar to the argument that monomial symmetric polynomials

$$\{m_\lambda|\lambda\in\mathcal{P}(k,n),k\geq 0\}$$

form a basis of Λ_n, one can prove that

$$\{A_\alpha(x_1,\dots,x_n)|d=(d_1,\dots,d_n)\in\mathbb{Z}^n,d_1>\cdots>d_n\geq 0\}$$

is a basis of A_n. Under the isomorphism F, they correspond to a basis

$$\{A_d(x_1,\dots,x_n)/A_\delta(x_1,\dots,x_n)|d=(d_1,\dots,d_n)\in\mathbb{Z}^n,d_1>\cdots>d_n\geq 0\}$$

is a basis of Λ_n.

We leave the proof of the following Lemma to the reader.

Lemma 7.2. *Suppose $d\in\mathbb{Z}^n$ satisfies*

$$d_1>\cdots>d_n\geq 0.$$

Define

$$\lambda_i=d_i-(n-i),$$

and

$$\lambda=(\lambda_1,\dots,\lambda_n),$$

i.e.,

$$\lambda=d-\delta.$$

Then λ is a partition of length $\leq n$.

Definition 7.4. For $\alpha\in\mathcal{P}(k,n)$, the *Schur polynomial associated to* λ is defined by:

$$s_\lambda(x_1,\dots,x_n)=\frac{A_{\lambda+\delta}(x_1,\dots,x_n)}{A_\delta(x_1,\dots,x_n)}.$$

From the above discussions, we have already given the proof of the following:

Proposition 7.2. *For any positive integer n,*

$$\{s_\lambda | \lambda \in \mathcal{P}(k,n)\}$$

is a basis of Λ_n^k.

7.4. **Generating series of Schur polynomials.** Consider the following generating series of Schur polynomials:

$$\begin{aligned} & S(x_1,\dots,x_n,y_1,\dots,y_n) \\ = & \sum_{d_1,\dots,d_n\geq 0} \frac{1}{\Delta(x_1,\dots,x_n)} \begin{vmatrix} x_1^{d_1} & \cdots & x_1^{d_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ x_n^{d_1} & \cdots & x_n^{d_n} \end{vmatrix} y_1^{d_1}\cdots y_n^{d_n}. \end{aligned}$$

Note for each $\lambda \in \mathcal{P}(k,n)$, the coefficient of

$$\prod_{j=1}^n y_j^{\lambda_j+n-j}$$

in $S(x_1,\dots x_n,y_1,\dots,y_n)$ is exactly $s_\lambda(x_1,\dots,x_n)$. It is not hard to see that

$$S(x_1,\dots,x_n,y_1,\dots,y_n) = \Delta(y_1,\dots,y_n)\sum_{l(\lambda)\leq n} s_\lambda(x)s_\lambda(y). \tag{18}$$

Theorem 7.3. *We have:*

$$S(x_1,\dots,x_n,y_1,\dots,y_n) = \frac{\Delta(y_1,\dots,y_n)}{\prod_{1\leq i,j\leq n}(1-x_iy_j)}. \tag{19}$$

Proof. By standard properties of the determinant we have:

$$\begin{aligned} & S(x_1,\dots,x_n,y_1,\dots,y_n) \\ = & \frac{1}{\Delta(x_1,\dots,x_n)} \begin{vmatrix} \sum_{d_1\geq 0}(x_1y_1)^{d_1} & \cdots & \sum_{d_n\geq 0}(x_1y_n)^{d_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \sum_{d_1\geq 0}(x_ny_1)^{d_1} & \cdots & \sum_{d_n\geq 0}(x_ny_n)^{d_n} \end{vmatrix} \\ = & \frac{1}{\Delta(x_1,\dots,x_n)} \begin{vmatrix} \frac{1}{1-x_1y_1} & \cdots & \frac{1}{1-x_1y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{1}{1-x_ny_1} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix}. \end{aligned}$$

The determinant can be evaluated as follows. Subtract the last row from the i-th row ($i > n$), and use common denominators. We get:

$$\begin{vmatrix} \frac{1}{1-x_1y_1} & \cdots & \frac{1}{1-x_1y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{1}{1-x_ny_1} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{y_1(x_1-x_n)}{(1-x_1y_1)(1-x_ny_1)} & \cdots & \frac{y_n(x_1-x_n)}{(1-x_1y_n)(1-x_ny_n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{y_1(x_{n-1}-x_n)}{(1-x_{n-1}y_1)(1-x_ny_1)} & \cdots & \frac{y_n(x_{n-1}-x_n)}{(1-x_{n-1}y_n)(1-x_ny_n)} \\ \frac{1}{1-x_ny_1} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{n-1}(x_i - x_n)}{\prod_{1\le j\le n}(1-x_ny_j)} \begin{vmatrix} \frac{y_1}{1-x_1y_1} & \cdots & \frac{y_n}{1-x_1y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{y_1}{1-x_{n-1}y_1} & \cdots & \frac{y_n}{1-x_{n-1}y_n} \\ 1 & \cdots & 1 \end{vmatrix}.$$

Now subtract the last column from the j-th column, use common denominators, and simply as above. We get:

$$\begin{vmatrix} \frac{1}{1-x_1y_1} & \cdots & \frac{1}{1-x_1y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{1}{1-x_ny_1} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix}$$

$$= \frac{\prod_{i=1}^{n-1}(x_i - x_n)(y_i - y_n)}{(1-x_ny_n)\prod_{1\le j\le n-1}(1-x_ny_j)(1-x_jy_n)} \begin{vmatrix} \frac{y_1}{1-x_1y_1} & \cdots & \frac{y_n}{1-x_1y_{n-1}} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{y_1}{1-x_{n-1}y_1} & \cdots & \frac{y_n}{1-x_{n-1}y_{n-1}} \end{vmatrix}.$$

Hence by induction one can show that

$$\begin{vmatrix} \frac{1}{1-x_1y_1} & \cdots & \frac{1}{1-x_1y_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{1}{1-x_ny_1} & \cdots & \frac{1}{1-x_ny_n} \end{vmatrix} = \frac{\Delta(x_1,\dots,x_n)\Delta(y_1,\dots,y_n)}{\prod_{1\le i,j\le n}(1-x_iy_j)}.$$

The proof of the Theorem is complete. ☐

Corollary 7.1. *We have*

$$\frac{1}{\prod_{i=1}^{n}(1-x_iy_j)} = \sum_{l(\lambda)\le n} s_\lambda(x)s_\lambda(y). \tag{20}$$

7.5. **Jacobi-Trudy formula.** It is very interesting to study the relationship between the basis given by Schur polynomials and the bases given by other types of symmetric polynomials. For example, there are *integers* $K_{\lambda\mu}$ (called *Kostka numbers*) such that

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu.$$

These numbers are interesting objects to study in algebraic combinatorics. There are also integers χ^λ_μ such that

$$p_\mu = \sum_\lambda \chi^\lambda_\mu s_\lambda, \tag{21}$$

$$s_\lambda = \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} p_\mu, \tag{22}$$

where

$$z_\mu = \prod_i i^{m_i(\mu)} m_i(\mu)!.$$

The integers $\{\chi^\lambda_\mu\}$ give the character table of the symmetric groups. For details, see §**??**.

In this subsection, we consider the relationship between Schur polynomials and elementary or complete symmetric polynomials.

Theorem 7.4. *(Jacobi-Trudy identities) For any $\lambda \in \mathcal{P}(k,n)$, the following identities hold:*

$$s_\lambda(x_1, \dots, x_n) \quad = \quad \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}. \tag{23}$$

Proof. Recall

$$\Delta(y_1, \dots, y_n) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^n y_j^{n-\sigma(j)},$$

$$\frac{1}{\prod_{i=1}^n (1 - x_i t)} = \sum_{m \geq 0} h_m(x_1, \dots, x_n) t^m.$$

Hence by (19), we have:

$$\begin{aligned} S(x,u) &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^n y_j^{n-\sigma(j)} \cdot \prod_{j=1}^n h_{m_j}(x) y_j^{m_j} \\ &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^n h_{m_j}(x) y_j^{m_j + n - \sigma(j)}. \end{aligned}$$

Consider the coefficients of

$$\prod_{j=1}^n y_j^{\lambda_j + n - j},$$

we get

$$\begin{aligned} s_\lambda(x) &= \sum_{\sigma \in S_n} (-1)^\sigma \prod_{j=1}^n h_{\lambda_j - j + \sigma(j)}(x) \\ &= \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}. \end{aligned}$$

This proves the Jacobi-Trudy identity. ∎

There is another Jacobi-Trudy identity:

$$s_\lambda(x_1, \ldots, x_n) \quad = \quad \det(e_{\lambda'_i - i + j}). \tag{24}$$

Its proof is more complicated. The interested reader can consult [7]. From (23) and (24), one easily see that

$$\omega(s_\lambda) = s_{\lambda'}.$$

8. Symmetric Functions

Most of the results above does not depend on the number of indeterminates. Hence one can consider the limit of infinitely many indeterminates. This leads to the space of symmetric functions.

8.1. **Space of symmetric functions.** For $m \geq n$, define $\rho_{m,n} : \Lambda_m \to \Lambda_n$ by

$$p(x_1, \dots, x_m) \mapsto p(x_1, \dots, x_n, 0, \dots, 0).$$

Define

$$\Lambda^k = \varprojlim_n \Lambda_n^k.$$

An element of Λ^k is a sequence

$$\{f_n(x_1, \dots, x_n) \in \Lambda_n^k\}$$

such that

$$f_n(x_1, \dots, x_{n-1}, 0) = f_{n-1}(x_1, \dots, x_{n-1}).$$

One can also regard it as a function in infinitely many variables:

$$f(x_1, \dots, x_n, \dots),$$

such that

$$f(x_1, \dots, x_n, 0, \dots) = f_n(x_1, \dots, x_n)$$

for all positive integer n.

Define

$$\Lambda = \oplus_{k \geq 0} \Lambda^k.$$

This is the space of all symmetric functions.

Most of the results in the preceding sections can be easily generalized to Λ, so we will leave their exact forms mostly to the reader. One can easily define m_λ, e_λ, h_λ, p_λ, and s_λ for infinitely many variables. They form bases of Λ. Furthermore, Λ is a graded ring with

$$p_t(\Lambda) = \prod_{i \geq 1} \frac{1}{1 - t^i}, \tag{25}$$

and one can take $\{e_1, \dots, e_n, \dots\}$, or $\{h_1, \dots, h_n, \dots\}$, or $\{p_1, \dots, p_n, \dots\}$ as free ring generators. Recall the Dedekind eta function is defined by:

$$\eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

8.2. **Three series expansions.**

Theorem 8.1. *The following identities hold:*

$$\frac{1}{\prod_{i,j \geq 1}(1 - x_i y_j)} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y) \tag{26}$$

$$= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) \tag{27}$$

$$= \sum_\lambda s_\lambda(x) s_\lambda(y). \tag{28}$$

Proof. Recall

$$\frac{1}{\prod_{i\geq 1}(1-x_iy)} = \sum_{r\geq 0} h_r(x)y^r,$$

hence we have

$$\begin{aligned}\frac{1}{\prod_{i,j\geq 1}(1-x_iy_j)} &= \prod_{j\geq 1}\sum_{r_j\geq 0} h_{r_j}(x)y_j^{r_j}\\ = \quad \sum\prod_j h_{r_j}(x)y_j^{r_j} &= \sum_\lambda h_\lambda(x)m_\lambda(y).\end{aligned}$$

This proves (26).

Recall

$$\begin{aligned}\frac{1}{\prod_{i\geq 1}(1-x_iy)} &= \exp\left(-\sum_{i\geq 1}\log(1-x_iy)\right)\\ = \quad \exp\left(\prod_{i\geq 1}\sum_{k\geq 1}\frac{1}{k}(x_iy)^k)\right) &= \exp\left(\sum_{k\geq 1}\frac{1}{k}p_k(x)y^k\right),\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{\prod_{i,j\geq 1}(1-x_iy_j)} &= \exp\left(\sum_{j\geq 1}\sum_{k\geq 1}\frac{1}{k}p_k(x)y_j^k\right)\\ = \quad \exp\left(\sum_{k\geq 1}\frac{1}{k}p_k(x)p_k(y)\right) &= \prod_{k\geq 1}\sum_{m_k\geq 0}\frac{p_k^{m_k}(x)p_k^{m_k}(y)}{m_k!k^{m_k}}\\ = \quad \sum_\lambda \frac{1}{z_\lambda}p_\lambda(x)p_\lambda(y).\end{aligned}$$

This proves (27).

One can prove (28) by taking $n\to\infty$ in (20). □

8.3. **Hermitian metric on Λ.** Regard x_i as real variables. The complex conjugation defines an involution on Λ.

We now define a scalar product on Λ by requiring

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu},$$

for all partitions λ and μ. Furthermore, $\langle\cdot,\cdot\rangle$ is required to have the following property:

$$\begin{aligned}\langle a_1f_1+a_2f_2,g\rangle &= a_1\langle f_1,g\rangle + a_2\langle f_2,g\rangle,\\ \langle f,b_1g_1+b_2g_2\rangle &= \bar{b}_1\langle f,g_1\rangle + \bar{b}_2\langle f,g_2\rangle.\end{aligned}$$

We will show show below this scalar product is actually a positive definite Hermitian metric.

Lemma 8.1. *For $k\geq 0$, let $\{u_\lambda\}$ and $\{v_\lambda\}$ be two bases of Λ^k, indexed by partitions of k. Then the following are equivalent:*

(a) $\langle u_\lambda, v_\mu\rangle = \delta_{\lambda,\mu}$, *for all* λ,μ.
(b) $\sum_\lambda u_\lambda(x)v_\mu(y) = \prod_{i,j\geq 1}(1-x_iy_j)^{-1}$.

Proof. Let $A = (a_{\lambda\rho})$ and $B = (b_{\mu\sigma})$ be two matrices such that

$$u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho, \qquad\qquad v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma.$$

Then we have

$$\langle u_\lambda, v_\mu \rangle = \sum_\rho a_{\lambda\rho} \bar{b}_{\mu\rho},$$

hence (a) is equivalent to

$$\sum_\rho a_{\lambda\rho} \bar{b}_{\mu\rho} = \delta_{\lambda\mu},$$

i.e.,

$$AB^* = I.$$

On the other hand,

$$\sum_\lambda u_\lambda(x)\bar{v}(y) = \sum_\lambda \sum_\rho \sum_\sigma a_{\lambda\rho} \bar{b}_{\lambda\sigma} h_\rho(x) m_\sigma(y),$$

$$\frac{1}{\prod_{i,j\geq 1}(1 - x_i y_j)} = \sum_\lambda h_\lambda(x) m_\lambda(y).$$

Hence (b) is equivalent to

$$\sum_\lambda a_{\lambda\rho} \bar{b}_{\lambda\sigma} = \delta_{\rho\sigma},$$

i.e.,

$$B^* A = I.$$

Therefore (a) and (b) are equivalent. □

Corollary 8.1. *We have*

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda,$$
$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

Proof. Straightforward consequences of (27), (28) and Lemma 8.1. □

Corollary 8.2. *The scalar product $\langle \cdot, \cdot \rangle$ is a positive definite Hermitian metric on Λ. Furthermore,*

$$\langle \omega(u), \omega(v) \rangle = \langle u, v \rangle$$

for $u, v \in \Lambda$.

9. Applications to Bosonic String Theory

In this section we will show that Λ admits a natural structure of a bosonic Fock space. We will also consider some consequences of this fact.

9.1. **Heisenberg algebra action.**

Definition 9.1. A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

such that

$$\begin{aligned} &[X,Y] = -[Y,X], \\ &[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]], \end{aligned}$$

for $X, Y, Z \in \mathfrak{g}$.

Let $\mathfrak{h}$ be the vector space spanned by $\{\alpha_n : n \in \mathbb{Z}\}$ and c. Define an antisymmetric bilinear form $[\cdot,\cdot] : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ by requiring:

$$[\alpha_m, \alpha_n] = m\delta_{m,-n}\alpha_m, \tag{29}$$

$$[c, \alpha_m] = 0, \tag{30}$$

$$[c, c] = 0, \tag{31}$$

for $m, n \in \mathbb{Z}$. It is easy to show that $(\mathfrak{h}, [\cdot,\cdot])$ is a Lie algebra. It is called the *(infinite) Heisenberg algebra.*

9.2. **Bosonic Fock space, creators, and annihilators.**

Definition 9.2. A linear representation of $\mathfrak{g}$ is a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}(V),$$

such that

$$\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X),$$

for all $X, Y \in \mathfrak{g}$. We will often write $\rho(X)v$ as Xv.

A highest weight representation of $\mathfrak{h}$ is a representation V of $\mathfrak{h}$, which has the following properties. There is a vector $|0\rangle \in V$, called the *vacuum vector*, such that

$$\alpha_n|0\rangle = 0$$

for $n \geq 0$; V is spanned by elements of the form $(n_1, \ldots, n_k > 0)$:

$$\alpha_{-n_1} \cdots \alpha_{-n_k}|0\rangle; \tag{32}$$

and c as by a multiplication by a constant. A highest weight representation is also called a bosonic Fock space.

The operators $\{\alpha_{-n}\}_{n>0}$ are said to be *creators*, and the operators $\{\alpha_n\}_{n\geq 0}$ *annihilators*. Physically, a vector of the form (32) represents a state which contains particles $\alpha_{-n_1}, \ldots, \alpha_{-n_k}$. The effect of the action of the operator α_{-n} on this vector is the addition of a particle α_{-n}, and the effect of the action of the operator α_n is the removal of a particle α_{-n}. This explains the terminology.

9.3. **Heisenberg algebra action on Λ.** We now show Λ has the natural structure of a bosonic Fock space. Define $\alpha_n : \Lambda \to \Lambda$ as follows:

$$\alpha_n f = \begin{cases} p_{-n}(x) \cdot f(x), & n < 0, \\ 0, & n = 0, \\ n \frac{\partial f}{\partial p_n}, & n > 0, \end{cases}$$

and let $c : \Lambda \to \Lambda$ be the identity map. It is straightforward to see that this defines an action of the Heisenberg algebra on Λ, for which 1 is the vacuum vector. Since Λ is spanned by $\{p_\lambda\}$, it is a bosonic Fock space.

Proposition 9.1. *With respect to the Hermitian metric on Λ, one has*

$$\alpha_n^* = \alpha_{-n},$$
$$c^* = c.$$

Proof. For the first identity, it suffices to prove the case of $n > 0$.

$$\begin{aligned} & \langle \alpha_n p_\lambda, p_\mu \rangle = \langle n \frac{\partial p_\lambda}{\partial p_n}, p_\mu \rangle = \langle n m_n(\lambda) p_n^{m_n(\lambda)-1} \prod_{i \neq n} p_i^{m_i(\lambda)}, \prod_i p_i^{m_i(\mu)} \rangle \\ = \quad & n m_n(\lambda) \delta_{m_n(\lambda)-1, m_n(\mu)} n^{m_n(\lambda)-1} (m_n(\lambda)-1)! \cdot \prod_{i \neq n} \delta_{m_i(\lambda), m_i(\mu)} i^{m_i(\lambda)} m_i(\lambda)! \\ = \quad & \delta_{m_n(\lambda), m_n(\mu)+1} n^{m_n(\lambda)} m_n(\lambda)! \cdot \prod_{i \neq n} \delta_{m_i(\lambda), m_i(\mu)} i^{m_i(\lambda)} m_i(\lambda)! \\ = \quad & \langle p_\lambda, p_n p_\mu \rangle = \langle p_\lambda, \alpha_{-n} p_\mu \rangle. \end{aligned}$$

The second identity is trivial. □

9.4. **Normal ordering and Virasoro algebra action on Λ.** The grading by degrees on Λ_n induces a natural grading on Λ:

$$\deg p_\lambda = |\lambda|.$$

This grading can be reformulated in terms of the operators α_n as follows. First consider the generalized Euler vector field:

$$L_0 = \sum_{n>0} n p_n \frac{\partial}{\partial p_n}.$$

It can be rewritten as:

$$L_0 = \sum_{n>0} \alpha_{-n} \alpha_n.$$

This expression is not symmetric since the sum is only taken over positive integers. It suggests one to consider the sum

$$K = \sum_{n \in \mathbb{Z}} \alpha_{-n} \alpha_n.$$

Unfortunately, one encounters an infinitity when one considers

$$K p_\lambda.$$

For example,

$$K|0\rangle = \sum_{n<0} \alpha_{-n} \alpha_n |0\rangle = \sum_{n>0} n|0\rangle.$$

To avoid such situations, physicists introduce the *normally ordered product* defined as follows:

$$: \alpha_{n_1} \cdots \alpha_{n_k} := \alpha_{n_{i_1}} \cdots \alpha_{n_{i_k}},$$

where $n_{i_1} \leq \cdots \leq n_{i_k}$ is a permutation of $n_1, \ldots, n_k$. It is then easy to see that

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \alpha_{-n} \alpha_n := \frac{1}{2} \sum_{k+l=0} : \alpha_k \alpha_l : .$$

Introduce

$$L_n = \frac{1}{2} \sum_{k+l=n} : \alpha_k \alpha_l : .$$

Remark 9.1. At first sight the definition of L_n involves an infinite sum and there might be an issue of convergence here. Since for any $v \in \Lambda$, $\alpha_n v = 0$ for n sufficiently large, hence $: \alpha_k \alpha_l v = 0$ for sufficiently large k or l, and so $L_n v$ actually involves only finitely many nonvanishing $: \alpha_k \alpha_l : v$. We will implicitly use this fact below.

We leave the proof of the following Lemma to the reader.

Lemma 9.1. *Let V be a vector space. Define $[\cdot, \cdot] : \mathrm{End}(V) \to \mathrm{End}(V)$ by*

$$[A, B] = AB - BA.$$

Then one has

$$\begin{aligned}
&[A, B] = -[B, A], \\
&[A, [B, C]] = [[A, B], C] + [B, [A, C]], \\
&[A, BC] = [A, B]C + B[A, C], \\
&[AB, C] = A[B, C] + [A, C]B.
\end{aligned}$$

From the definition of the normally ordered product and the commutation relation (29), one easily verifies the following:

Lemma 9.2. *We have*

$$: \alpha_m \alpha_n := \begin{cases} \alpha_m \alpha_n - m\delta_{m,-n}\,\mathrm{id}, & m > 0, n < 0, \\ \alpha_m \alpha_n, & \textit{otherwise.} \end{cases} \tag{33}$$

In particular,

$$[A, : \alpha_m \alpha_n :] = [A, \alpha_m \alpha_n],$$

for $A \in \mathrm{End}(\Lambda)$, $m, n \in \mathbb{Z}$.

Theorem 9.1. *One has the following commutation relations:*

$$\begin{aligned}
&[\alpha_m, L_n] = m\alpha_{m+n}, \\
&[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}.
\end{aligned}$$

Proof. By Lemma 9.2 we have

$$\begin{aligned}
[\alpha_m, L_n] &= \frac{1}{2}\sum_{k+l=n}[\alpha_m, :\alpha_k\alpha_l:] = \frac{1}{2}\sum_{k+l=n}[\alpha_m, \alpha_k\alpha_l] \\
&= \frac{1}{2}\sum_{k+l=n}([\alpha_m,\alpha_k]\alpha_l + \alpha_k[\alpha_m,\alpha_l]) \\
&= \frac{1}{2}\sum_{k+l=n}(m\delta_{m,-k}\alpha_l + \alpha_k \cdot m\delta_{m,-l}) \\
&= m\alpha_{m+n}.
\end{aligned}$$

To prove the second identity we assume $m \geq n$ without loss of generality.

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}[:\alpha_{m-k}\alpha_k:, L_n] \\
&= \frac{1}{2}\sum_{k>0}[\alpha_{m-k}\alpha_k, L_n] + \frac{1}{2}\sum_{k>0}[\alpha_{-k}\alpha_{m+k}, L_n] \\
&= \frac{1}{2}\sum_{k>0}(\alpha_{m-k}[\alpha_k, L_n] + [\alpha_{m-k}, L_n]\alpha_k \\
&\quad +\alpha_{-k}[\alpha_{m+k}, L_n] + [\alpha_{-k}, L_n]\alpha_{m+k}) \\
&= \frac{1}{2}\sum_{k>0}(\alpha_{m-k}\cdot k\alpha_{k+n} + (m-k)\alpha_{m+n-k}\alpha_k \\
&\quad +\alpha_{-k}\cdot(m+k)\alpha_{m+n+k} + (-k)\alpha_{-k+n}\alpha_{m+k}) \\
&= \frac{1}{2}\sum_{k>0}(k:\alpha_{m-k}\alpha_{k+n}: -k:\alpha_{m+k}\alpha_{-k+n}: \\
&\quad +(m-k):\alpha_{m+n-k}\alpha_k: +(m+k):\alpha_k\alpha_{-k}:) \\
&\quad +\frac{1}{2}\sum_{k=1}^{m-1}k(m-k)\delta_{m,-n}\,\mathrm{id} \\
&= \frac{1}{2}\sum_{p+q=m+n}((q-n):\alpha_p\alpha_q: +(-q+m):\alpha_p\alpha_q:) + \frac{m^3-m}{24}\delta_{m,-n}\,\mathrm{id} \\
&= (m-n)L_0 + \frac{m^3-m}{24}\delta_{m,-n}\,\mathrm{id}\,.
\end{aligned}$$

□

9.5. **Vertex operator.** Consider the generating series of operators α_n:

$$\alpha(z) = \sum_{n\in\mathbb{Z}}\alpha_n z^{-n-1}.$$

This is a "field of operators". Integrating once, one gets another series:

$$Y(z) = C + a_0 \ln z + \sum_{n>0}\frac{\alpha_{-n}}{n}z^n - \sum_{n>0}\frac{\alpha_n}{n}z^{-n}.$$

For simplicity, we will take $C = a_0 = 0$. Now note

$$\sum_{n>0}\frac{\alpha_{-n}}{n}z^n|0\rangle = \sum_{n>0}\frac{p_n}{n}z^n = P(z) = \log H(z).$$

This suggests one to consider

$$\exp\left(\sum_{n>0}\frac{\alpha_{-n}}{n}z^n\right),$$

its "adjoint"

$$\exp\left(\sum_{n>0}-\frac{\alpha_n}{n}z^{-n}\right),$$

and the field:

$$X(z) =: \exp Y(z) := \exp\left(\sum_{n>0}\frac{\alpha_{-n}}{n}z^n\right)\exp\left(\sum_{n>0}-\frac{\alpha_n}{n}z^{-n}\right).$$

This is the vertex operator in free bosonic string theory. Write:

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n}.$$

It is not hard to see that

$$X(z)|0\rangle = H(z).$$

I.e.,

$$X_{-n}|0\rangle = h_n.$$

In general, one has [4]

Theorem 9.2. *For any partition $\lambda_1 \geq \cdots \geq \lambda_n > 0$, one has*

$$X_{-\lambda_1}\cdots X_{-\lambda_n}|0\rangle = s_\lambda.$$

Appendix A. Basics of Free Bosonic String Theory: The Physical Approach

In this section we sketch some basics of the physical theory of the free bosonic strings.

A.1. **Lagrangian of free boson on cylinder.** The trajectory of a closed string moving in the Minkowski space $\mathbb{R}^{3,1}$ is a cylinder $S^1 \times \mathbb{R}$, hence it can be described by a smooth map

$$f : \mathbb{R}^1 \times S^1 \to \mathbb{R}^4,$$

or equivalently by four functions:

$$\varphi^i : \mathbb{R}^1 \times S^1 \to \mathbb{R}.$$

Take linear coordinate x^0 on $\mathbb{R}^1$. Let

$$\{e^{\sqrt{-1}x^1} : 0 \leq x^1 < 2\pi\}$$

be the set of all the points on S^1. Endow the cylinder with a Riemannian metric $g = (dx^0)^2 + (dx^2)^2$. For simplicity of presentation, we will deal with only one component of the map f, and denote it by φ. The Lagrangian is given by:

$$L(\varphi) = \frac{1}{2} \int_{\mathbb{R}^1 \times S^1} ((\partial_{x^0}\varphi)^2 + (\partial_{x^1}\varphi)^2) dx^0 dx^1.$$

A.2. **Equation of motion and its solutions.** By calculus of variation one can obtain the equation of motion of the bosonic string as follows.

$$\begin{aligned}
& \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} L(\varphi + \epsilon\psi) \\
= & \left.\frac{1}{2}\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_{\mathbb{R}^1 \times S^1} ((\partial_{x^0}\varphi + \epsilon\partial_{x^0}\psi)^2 + (\partial_{x^1}\varphi + \epsilon\partial_{x^1}\psi)^2) dx^0 dx^1 \\
= & \int_{\mathbb{R}^1 \times S^1} (\partial_{x^0}\varphi\partial_{x^0}\psi + \partial_{x^1}\varphi\partial_{x^1}\psi) dx^0 dx^1 \\
= & -\int_{\mathbb{R}^1 \times S^1} (\partial^2_{x^0}\varphi + \partial^2_{x^1}\varphi)\psi dx^0 dx^1.
\end{aligned}$$

Hence

$$\partial^2_{x^0}\varphi + \partial^2_{x^1}\varphi = 0.$$

By separation of variables, we get the following form of solutions:

$$\begin{aligned}
\varphi(x^0, x^1) &= a + bx^0 - \sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{a_n}{n} e^{-n(x^0 + \sqrt{-1}x^1)} + \frac{\tilde{a}_n}{n} e^{-n(x^0 - \sqrt{-1}x^1)} \right) \\
&= a + a_0 \ln z + \tilde{a}_0 \ln \bar{z} - \sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{a_n}{n} z^{-n} + \frac{\tilde{a}_n}{n} \bar{z}^{-n} \right),
\end{aligned}$$

where

$$z = x^0 + \sqrt{-1}x^1, \qquad \bar{z} = x^0 - \sqrt{-1}x^1.$$

A field is said to be *chiral* if it is holomorphic. So the chiral part of φ is

$$\varphi(z) = a + a_0 \ln z - \sum_{n \in \mathbb{Z} - \{0\}} \frac{a_n}{n} z^{-n}.$$

In particular,

$$\partial_z \varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}.$$

A.3. **Energy-momentum tensor.** This is defined in this case by

$$T^{ij} = \frac{\partial \mathcal{L}}{\partial(\partial_{x^i}\varphi)} \cdot \partial_{x^j}\varphi - \delta_{ij}\mathcal{L}, \tag{34}$$

where

$$\mathcal{L} = \frac{1}{2}((\partial_{x^0}\varphi)^2 + (\partial_{x^1}\varphi)^2)$$

is the Lagrangian density. It is easy to see that

$$\begin{aligned} T^{00} &= -T^{11} = \frac{1}{2}((\partial_{x^0}\varphi)^2 - (\partial_{x^1}\varphi)^2), \\ T^{01} &= T^{10} = \partial_{x^0}\varphi\partial_{x^1}\varphi. \end{aligned}$$

In particular,

$$T^{00} + T^{11} = 0.$$

Proposition A.1. *For φ satisfying the equation of motion, we have*

$$\partial_{x^i} T^{ij} = 0.$$

Proof. For $j = 0$ we have

$$\begin{aligned} & \partial_{x^i} T^{i0} \\ = \; & \frac{1}{2}\partial_{x^0}((\partial_{x^0}\varphi)^2 - (\partial_{x^1}\varphi)^2) + \partial_{x^1}(\partial_{x^0}\varphi\partial_{x^1}\varphi) \\ = \; & (\partial_{x^0}^2\varphi)(\partial_{x_0}\varphi) - (\partial_{x^0}\partial_{x^1}\varphi)(\partial_{x^1}\varphi) + (\partial_{x^1}\partial_{x^0}\varphi)(\partial_{x^1}\varphi) + (\partial_{x^0}\varphi)(\partial_{x_1}^2\varphi) \\ = \; & 0. \end{aligned}$$

The case of $j = 1$ is similar. □

Recall

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_{x^0} - \sqrt{-1}\partial_{x^1}), & \partial_{\bar{z}} &= \frac{1}{2}(\partial_{x^0} + \sqrt{-1}\partial_{x^1}), \\ dz &= dx^0 + \sqrt{-1}dx^1, & d\bar{z} &= dx^0 - \sqrt{-1}dx^1. \end{aligned}$$

Write

$$T = T^{ij}dx^i dx^j = T^{zz}dzdz + T^{z\bar{z}}dzd\bar{z} + T^{\bar{z}z}d\bar{z}dz + T^{\bar{z}\bar{z}}d\bar{z}d\bar{z}..$$

By straightforward calculations, one finds

$$\begin{aligned} T^{zz} &= \frac{1}{4}(T^{00} - T^{11} + \frac{1}{\sqrt{-1}}T^{01} + \frac{1}{\sqrt{-1}}T^{10}) = (\partial_z\varphi)^2, \\ T^{z\bar{z}} &= \frac{1}{4}(T^{00} + T^{11} - \frac{1}{\sqrt{-1}}T^{01} + \frac{1}{\sqrt{-1}}T^{10}) = 0, \\ T^{\bar{z}z} &= \frac{1}{4}(T^{00} + T^{11} - \frac{1}{\sqrt{-1}}T^{01} + \frac{1}{\sqrt{-1}}T^{10}) = 0, \\ T^{\bar{z}\bar{z}} &= \frac{1}{4}(T^{00} - T^{11} - \frac{1}{\sqrt{-1}}T^{01} - \frac{1}{\sqrt{-1}}T^{10}) = (\partial_{\bar{z}}\varphi)^2. \end{aligned}$$

When the field φ is chiral, the nonvanishing component of T is

$$T^{zz} = (\partial_z\varphi)^2.$$

A.4. **Quantization and the bosonic Fock space.** Upon quantization, coefficients a and a_n becomes operators on a Hilbert space. For simplicity, we first take a and a_0 to be the zero operators. For $n < 0$, a_n is a creator; for $n > 0$, a_n is an annihilator. The Hilbert space B in concern contains a vacuum vector $|0\rangle$, i.e.,

$$a_n|0\rangle = 0$$

for $n \geq 0$, and B has an orthogonal basis of the form

$$\{a_{-n_1} \cdots a_{-n_k}|0\rangle : n_1, \dots, n_k > 0, k \geq 0\}.$$

Furthermore,

$$[a_m, a_n] = m\delta_{m,-n} \tag{35}$$

on B.

One can also consider another similar space $\widetilde{B}$ on which $\{a_n : n \geq 0\}$ are annihilators, and $\{a, a_n : n < 0\}$ are creators, and

$$[a_0, a] = 1.$$

A.5. **Vacuum expectation values and Wick Theorem.** For an operator A on B or $\widetilde{B}$, define the *vacuum expectation value (vev)* of A by

$$\langle A\rangle = \langle 0|A|0\rangle.$$

Theorem A.1. *(Wick Theorem, Version I) Let $k_1, \dots, k_m, l_1, \dots, l_n$ be positive integers, then*

$$\langle a_{k_1} \cdots a_{k_m} a_{-l_1} \cdots a_{-l_n}\rangle = 0$$

unless $m = n$, and

$$\langle a_{k_1} \cdots a_{k_n} a_{-l_1} \cdots a_{-l_n}\rangle = \sum_{\sigma\in S_n} \prod_{i=1}^{n} \langle a_{k_i} a_{-l_{\sigma(i)}}\rangle = \sum_{\sigma\in S_n} \prod_{i=1}^{n} k_i \delta_{k_i,l_{\sigma(i)}}.$$

Proof. Easy consequence of (35). □

For a partition μ of length l, let

$$a_\mu = a_{\mu_1} \cdots a_{\mu_l}, \qquad a_{-\mu} = a_{-\mu_1} \cdots a_{-\mu_l}.$$

Then we have:

Corollary A.1. *For two partitions μ and ν we have:*

$$\langle a_\mu a_{-\nu}\rangle = z_\mu \delta_{\mu\nu}.$$

A.6. **n-point functions.** The vev

$$\langle \varphi(z_1) \cdots \varphi(z_n)\rangle$$

is called the *n-point function.*

Proposition A.2. *On the Fock space $\widetilde{B}$, we have*

$$\langle \varphi(z)\rangle = 0, \tag{36}$$

$$\langle \varphi(z)\varphi(w)\rangle = \ln(z - w) = \ln z - \sum_{n>0} \left(\frac{w}{z}\right)^n. \tag{37}$$

Proof. The first identity is trivial. The second identity is proved as follows.

$$\begin{aligned}
& \langle\varphi(z)\varphi(w)\rangle \\
= & \left\langle \left(a + a_0 \ln z - \sum_{n\in\mathbb{Z}-\{0\}} \frac{a_n}{n} z^{-n}\right)\left(a + a_0 \ln w - \sum_{m\in\mathbb{Z}-\{0\}} \frac{a_m}{m} z^{-m}\right)\right\rangle \\
= & \ln z - \sum_{n>0} \frac{1}{n} z^{-n} w^n = \ln z + \ln(1 - \frac{w}{z}) \\
= & \ln(z-w).
\end{aligned}$$

□

We will mostly be concerned with the vevs on B. Let

$$\beta(z) = \partial_z \varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}.$$

Proposition A.3. *On the Fock space B, we have*

$$\langle\beta(z)\rangle = 0, \tag{38}$$

$$\langle\beta(z)\beta(w)\rangle = z^{-2}\sum_{n\geq 1} n(w/z)^{n-1} = \frac{1}{(z-w)^2}. \tag{39}$$

Proof. The first identity is trivial. The second identity is proved as follows.

$$\begin{aligned}
\langle\beta(z)\beta(w)\rangle &= \left\langle \left(\sum_{n\in\mathbb{Z}} a_n z^{-n-1}\right)\left(\sum_{m\in\mathbb{Z}} a_m z^{-m-1}\right)\right\rangle \\
&= \sum_{n>0} n z^{-n-1} w^{n-1} = z^{-2}\sum_{n\geq 1} n(w/z)^{n-1} \\
&= \frac{1}{(z-w)^2}.
\end{aligned}$$

□

A.7. **Operator product expansions.** We begin with an example. Clearly $\beta(z)\beta(w)$ and $:\beta(z)\beta(w):$ are different. We now consider their difference:

$$\begin{aligned}
& \beta(z)\beta(w) - :\beta(z)\beta(w): \\
= & \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \cdot \sum_{m\in\mathbb{Z}} a_m w^{-m-1} - :\sum_{n\in\mathbb{Z}} a_n z^{-n-1} \cdot \sum_{m\in\mathbb{Z}} a_m w^{-m-1}: \\
= & \sum_{n>0} n z^{-n-1} w^{n-1} = \frac{1}{(z-w)^2}.
\end{aligned}$$

Hence

$$\beta(z)\beta(w) = \frac{1}{(z-w)^2} + :\beta(z)\beta(w):.$$

Note when $z \to w$, the first term is singular, while the second term is regular in the sense that it has the limit $:\beta(w)\beta(w):$. We often rewrite it as

$$\beta(z)\beta(w) = \frac{1}{(z-w)^2} + :\beta(z)\beta(w):. \tag{40}$$

An expression of this form is often called an *operator product expansion (OPE)*. See [5] for a nice mathematical treatment of the OPEs, in particular, the proof of the following important result.

Theorem A.2. *(Wick Theorem for OPEs) Let* $\{a^1(z), \dots, a^M(z), b^1(z), \dots, b^N(z)\}$ *be a collection of fields such that the singular parts* $[a^i b^j]$ *of* $a^i(z)b^j(w)$ *are multiples of the identity operators. Then we have the following OPE:*

$$: a^1(z) \cdots a^M(z) :: b^1(w) \cdots b^N(w) :$$

Then one has:

$$\begin{aligned} & : a^1(z) \cdots a^M(z) :: b^1(w) \cdots b^N(w) : \\ = & \sum_{s=0}^{\min(M,N)} [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] : a^1(z) \cdots a^M(z) b^1(w) \cdots b^N(w) :_{(i_1, \cdots, i_s; j_1, \cdots, j_s)}, \end{aligned}$$

where the subscript $(i_1, \cdots, i_s; j_1, \cdots, j_s)$ *means that the fields* $a^{i_1}(z), \cdots, a^{i_s}(z)$, $b^{j_1}(w), \cdots, b^{j_s}(w)$ *are removed.*

The n-point function can be computed by the Wick Theorem. Recall the energy moment field is

$$T(z) = \frac{1}{2} : \beta(z)\beta(z) := \frac{1}{2} : \partial_z \varphi \partial_z \varphi(z) : .$$

We are also interested in

$$\Phi(z) = \frac{1}{3!} : \beta(z)^3 := \frac{1}{6} : (\partial_z \varphi(z))^3 : .$$

Using the Wick Theorem, it is straightforward to get the following:

Proposition A.4. *We have*

$$T(z)\beta(w) \sim \frac{\beta(w)}{(z-w)^2} + \frac{\partial_w \beta(w)}{z-w}, \tag{41}$$

$$T(z)T(w) \sim \frac{\partial_w T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{1/2}{(z-w)^4}, \tag{42}$$

$$\Phi(z)\beta(w) \sim \frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \tag{43}$$

Proof. One has

$$T(Z)\beta(w) = \frac{1}{2} : \beta(z)^2 : \beta(w) \sim \frac{\beta(z)}{(z-w)^2} \sim \frac{\beta(w)}{(z-w)^2} + \frac{\partial_w \beta(w)}{z-w}.$$

The other two OPEs can be obtained in the same fashion. □

A.8. **Vertex operator.** The vertex operators

$$V(z) =: e^{\varphi(z)} := \exp\left(\sum_{n>0} \frac{a_{-n}}{n} z^n\right) \exp\left(\sum_{n>0} -\frac{a_n}{n} z^n\right)$$

and

$$\overline{V}(z) = \exp\left(\sum_{n>0} \frac{a_{-n}}{n} z^n\right) e^a z^{a_0} \exp\left(\sum_{n>0} -\frac{a_n}{n} z^n\right)$$

are introduced by string theorists (cf. e.g. [3]).

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