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| MAT 4220 | NAME: ______________ |
| EXAM 1 | STUDENT ID #: ______________ |

## Problems

1. Calculate the limit $\lim_{x\to\infty} \sqrt{x^4+4x^2+1} - x^2$.

2. If $x^2 + y^2 = 1$, then show that $y'' = -1/y^3$.

3. Determine the Maclaurin polynomial of the third degree, $T_3(x)$, for $f(x) = \tan(x)$.

4. Let $\chi(x)$ be the *characteristic function* of the rationals; i.e., $\chi(x) = \left\{ \begin{array}{cl} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{array} \right\}$. Explain why $\chi$ is nowhere differentiable.

5. Derivatives possess the *Intermediate Value Property*; that is,

   **Theorem 1** *If $f$ is differentiable on $[a,b]$ and $m$ is any value between $f'(x_1)$ & $f'(x_2)$ for $[x_1,x_2] \subseteq [a,b]$, then there exists a point $c \in (x_1,x_2)$ for which $f'(c) = m$.*

   Show $f(x) = |x|$ does not have the *IVP*. Explain why this does not contradict the theorem.

## Proofs

6. **Theorem 2 (Chain Rule)** *Let $S, T \subseteq \mathbb{R}$ be open intervals with $f : S \to T$ and $g : T \to \mathbb{R}$. Suppose that $f$ is differentiable at $x = a \in S$ and $g$ is differentiable at $t = f(a) \in T$. Then $g \circ f$ is differentiable at $x = a$ and*

$$(g \circ f)'(a) = g'(f(a))\, f'(a)$$

7. **Theorem 3 (Rolle's Theorem)** *Suppose that*

   *(a)* *$f$ is continuous on the closed interval $[a,b]$,*
   *(b)* *$f$ is differentiable on the open interval $(a,b)$, and*
   *(c)* $f(a) = f(b)$.

   *Then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.*

8. **Definition 1** *A function $f : D \to \mathbb{R}$ is* Lipschitz $\alpha$, *written $f \in Lip_\alpha(D)$, if and only if there exists a constant $K \in \mathbb{R}$ such that*

$$|f(x_1) - f(x_2)| \leq K\,|x_1 - x_2|^\alpha$$

   *for all $x_1, x_2 \in D$.*

   **Theorem 4** *Let $f$ be a function defined on $I = [a,b]$ and let $\alpha > 1$. Then $f \in Lip_\alpha(I)$ if and only if $f$ is a constant function.*