MAT 5620, Analysis II

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Fall, 2011

Vector Calculus

Vector Space Axioms

A set $\mathcal{V} = \{\vec{v}\}$ with addition $+$ and scalar multiplication $\cdot$ with scalars from a field F is a *vector space over* F when

1. $\langle \mathcal{V}, + \rangle$ is an Abelian group.
2.
   - scalar multiplication distributes over vector addition
   - scalar addition distributes over scalar multiplication
   - multiplication of scalars ‘associates’ with scalar multiplication

Recall:

- The *norm* (magnitude) of a vector $\vec{u}$ is $\|\vec{u}\| = \sqrt{\sum u_i^2}$
- The *direction vector* of $\vec{u}$ is $(1/\|u\|) \cdot \vec{u}$

Definition (Dot Product in $\mathbb{R}^n$ over $\mathbb{R}$)

Dot Product $\quad \vec{u} \cdot \vec{v} = \sum u_i \cdot v_i = \|\vec{u}\| \, \|\vec{v}\| \, \cos(\angle \overline{uv})$

Dot Product

Proposition (Dot Product Properties)

Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. Then

1. $\angle\overline{uv} = \cos^{-1}\left[\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right]$ *angle between vectors*
2. $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$ *Cauchy-Bunyakovsky-Schwarz inequality*
3. $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ *Triangle inequality; (cf. Minkowski's inequality)*
4. $\mathrm{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$ *(orthogonal) projection of $\vec{u}$ onto $\vec{v}$*

Cross Product

Definition

- Let $\vec{u}$ and $\vec{v} \in \mathbb{R}^3$; set e_1, e_2, e_3 to be std basis vectors. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Let $\vec{u_1}$ to $\vec{u}_{n-1} \in \mathbb{R}^n$, $n \geq 3$; let $\{e_n\} = \{$std basis vectors$\}$. Then

$$\times(\vec{u_1}, \dots, \vec{u}_{n-1}) = \begin{vmatrix} e_1 & e_2 & \dots & e_n \\ u_{1,1} & u_{1,2} & \dots & u_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \dots & u_{n-1,n} \end{vmatrix}$$

Cross Product Properties

Proposition (Cross Product Properties in $\mathbb{R}^3$)

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be in $\mathbb{R}^3$. Then

1. $\angle\overline{uv} = \sin^{-1}\left[\dfrac{\|\vec{u}\times\vec{v}\|}{\|\vec{u}\|\,\|\vec{v}\|}\right]$ *angle between vectors*
2. $\|\vec{u}\times\vec{v}\| \le \|\vec{u}\|\,\|\vec{v}\|$
3. $\vec{u}\times\vec{v} = -\,\vec{v}\times\vec{u}$ *area of* $[\vec{u},\vec{v}] = \|\vec{u}\times\vec{v}\|$
4. $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w} = \vec{v}\cdot(\vec{w}\times\vec{u})$
5. $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$; *volume of* $[\vec{u},\vec{v},\vec{w}] = |\vec{u}\cdot(\vec{v}\times\vec{w})|$

Parametric Equations

Definition (Parametrization)

Suppose $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$, and $h: D \to \mathbb{R}$. Then

$$\gamma(t) = (f(t), g(t), h(t))$$

for $t \in D$ is a *curve (spacecurve)* in $\mathbb{R}^3$. The fcns f, g, and h are *parametric equations* for γ, or a *parametrization of* γ.

Examples

1. The line segment L from $\vec{u}$ to $\vec{w}$ can be parametrized as

$$L(t) = \vec{u} + (\vec{w} - \vec{u}) \cdot t, \qquad t \in [0, 1]$$

2. Γ given by `f:=t->⟨cos(t),sin(t)*cos(t),t*(1-t)⟩` for $t \in [0, 3\pi]$.

```
animate(spacecurve,[f(t),t=0..3*Pi*k,
thickness=2],k=0..1,axes=frame,color=black,frames=30)
```

Continuous Spacecurves

Definition

Let $\mathcal{I} = [a, b] \subseteq \mathbb{R}$. A curve γ is

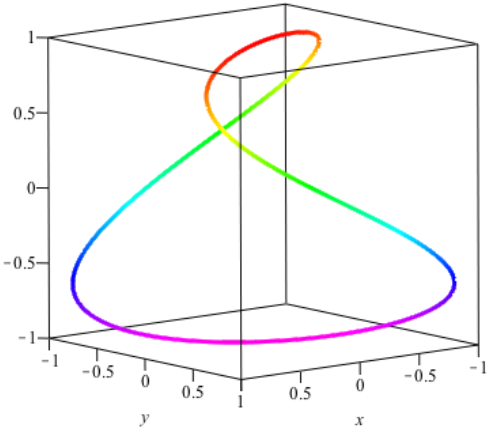
- *continuous (on $\mathcal{I}$)* if γ can be parametrized with components that are continuous on $\mathcal{I}$.
- *smooth (on $\mathcal{I}$)* if γ's parametric components are continuously differentiable on $\mathcal{I}$, and ${f'}^2 + {g'}^2 + {h'}^2 > 0$ for all $t \in (a, b)$.
- *piecewise smooth (on $\mathcal{I}$)* if $[a, b]$ can be partitioned into a finite number of subintervals on which γ is smooth.

Note: Smooth $\equiv$ a particle moving parametrically along the curve doesn't change direction abruptly, stop mid-curve, or reverse.

Theorem

If $\gamma(t) = (f(t), g(t))$ is smooth on $[a, b]$, then tangent slope at $P_0 = (x, y)$ is given by $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$ when $\dfrac{dx}{dt} \neq 0$.

A Smooth Closed Curve



$\Gamma(t) = (\sin(2t), \sin(t), \cos(t))$ for $t \in [0, 2\pi]$

$$\Gamma(0) = \Gamma(2\pi)$$

Lines in $\mathbb{R}^3$

Theorem (The Line Forms Here Thm)

A line ℓ passing through $P_0 = (x_0, y_0, z_0)$, parallel to $\vec{u} = (a, b, c) \neq \vec{0}$ has

vector form: $\ell(t) = P_0 + t\,\vec{u}$, $t \in \mathbb{R}$

parametric form: $\ell(t) = (x_0 + at,\ y_0 + bt,\ z_0 + ct)$, $t \in \mathbb{R}$

symmetric form: $\dfrac{x(t) - x_0}{a} = \dfrac{y(t) - y_0}{b} = \dfrac{z(t) - z_0}{c}$

Consider...

Let $P_0 = (1, 2, 4)$ and direction $\vec{u} = (1, 2, -1)$.

1. $\ell_1(t) = (1 + t,\ 2 + 2t,\ 4 - t)$ $\qquad \vec{u} = (1, 2, -1)$
2. $\ell_2(s) = \left(1 + \frac{1}{\sqrt{6}}\,s,\ 2 + \frac{2}{\sqrt{6}}\,s,\ 4 - \frac{1}{\sqrt{6}}\,s\right)$ $\qquad \vec{w} = \frac{1}{\sqrt{6}}(1, 2, -1)$

Planes in $\mathbb{R}^3$

Theorem (The Plane, the Plane)

A plane P passing through $P_0 = (x_0, y_0, z_0)$, normal to $\vec{u} = (a, b, c) \neq \vec{0}$ is $P = \{\vec{X}\}$ s.t.

vector form: $\vec{u} \cdot \left(\vec{X} - P_0\right) = 0$

parametric form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

A plane P passing through $P_0 = (x_0, y_0, z_0)$, containing two vectors $\vec{u}$ and $\vec{w}$ is $P = \{\vec{X}\}$ s.t.

cross product form: $(\vec{u} \times \vec{w}) \cdot \left(\vec{X} - P_0\right) = 0$

Problem

1. *Find a plane containing the three points* $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

Quadric Surfaces

Standard Forms of Quadric Surfaces

sphere:	$x^2 + y^2 + z^2 = r^2$
ellipsoid:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
elliptic paraboloid:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$
hyperbolic paraboloid:	$\frac{x^2}{a^2} - \frac{y^2}{b^2} + z = 0$
elliptic cone:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 = 0$
hyperboloid of 1 sheet:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = +1$
hyperboloid of 2 sheets:	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

Quadric Surfaces Reformed

Almost Standard Forms of Quadric Surfaces

sphere: $\rho x^2 + \rho y^2 + \rho z^2 = 1$

ellipsoid: $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$

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elliptic paraboloid: $\alpha x^2 + \beta y^2 - z = 0$

hyperbolic paraboloid: $\alpha x^2 - \beta y^2 + z = 0$

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elliptic cone: $\alpha x^2 + \beta y^2 - z^2 = 0$

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hyperboloid of 1 sheet: $\alpha x^2 + \beta y^2 - \gamma z^2 = +1$

hyperboloid of 2 sheets: $\alpha x^2 + \beta y^2 - \gamma z^2 = -1$

Vector-Valued Functions

Notation

The *standard basis vectors* in $\mathbb{R}^3$ are

$$\langle 1,0,0\rangle = e_1 = \mathbf{i}, \qquad \langle 0,1,0\rangle = e_2 = \mathbf{j}, \qquad \langle 0,0,1\rangle = e_3 = \mathbf{k}$$

If f, g, $h : D \to \mathbb{R}$ are real functions, then $\vec{r} : D \to \mathbb{R}^3$ given by

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a *vector-valued function* with components f, g, and h.

Definition

Let $\vec{r} : D \to \mathbb{R}^3$ have components f, g, and h, and let t_0 be an accumulation point of D. Then

$$\lim_{t\to t_0} \vec{r}(t) = \vec{L} = L_f\mathbf{i} + L_g\mathbf{j} + L_h\mathbf{k}$$

iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t \in D)$ if $0 < |t - t_0| < \delta$, then $\|\vec{r}(t) - \vec{L}\| < \epsilon$.

Vector-Valued Function Limits

Theorem (Limits Work)

$$\lim_{t\to t_0} \vec{r}(t) = L_f\mathbf{i} + L_g\mathbf{j} + L_h\mathbf{k}$$

$$\Longleftrightarrow$$

$$\lim_{t\to t_0} f(t) = L_f \;\wedge\; \lim_{t\to t_0} g(t) = L_g \;\wedge\; \lim_{t\to t_0} h(t) = L_h$$

Proof (key inequality).

$$|a| \underset{(\Leftarrow)}{\leq} \sqrt{a^2+b^2+c^2} = \|(a,b,c)\| \underset{(\Rightarrow)}{\leq} |a|+|b|+|c|$$

$\square$

Algebra of Vector-Valued Function Limits

Theorem (Algebra of Vector-Valued Limits)

Suppose $\vec{u}, \vec{w} : D \to \mathbb{R}^n$, $k : D \to \mathbb{R}$, $c \in \mathbb{R}$, *and* $t_0 \in D'$. *Then*

$$\lim_{t \to t_0} [\vec{u} \pm \vec{w}] = \left[\lim_{t \to t_0} \vec{u}\right] \pm \left[\lim_{t \to t_0} \vec{w}\right] \tag{1}$$

$$\lim_{t \to t_0} [c\vec{u}] = c\left[\lim_{t \to t_0} \vec{u}\right] \tag{2}$$

$$\lim_{t \to t_0} [k\vec{u}] = \left[\lim_{t \to t_0} k\right]\left[\lim_{t \to t_0} \vec{u}\right] \tag{3}$$

$$\lim_{t \to t_0} [\vec{u} \cdot \vec{w}] = \left[\lim_{t \to t_0} \vec{u}\right] \cdot \left[\lim_{t \to t_0} \vec{w}\right] \tag{4}$$

$$\lim_{t \to t_0} [\vec{u} \times \vec{w}] = \left[\lim_{t \to t_0} \vec{u}\right] \times \left[\lim_{t \to t_0} \vec{w}\right] \tag{5}$$

Continuity of Vector-Valued Functions

Definition (Continuity)

A function $\vec{r}(t)$ is *continuous* at $t_0 \in D$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t \in D)$ if $|t - t_0| < \delta$, then $\|\vec{r}(t) - \vec{r}(t_0)\| < \epsilon$.

Proposition

1. *A function $\vec{r}(t)$ is continuous at an accumulation point $t_0 \in D$ iff*

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$

2. *A function $\vec{r}(t)$ is uniformly continuous on $E \subseteq D$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t_1, t_2 \in E)$ if $|t_1 - t_2| < \delta$, then $\|\vec{r}(t_1) - \vec{r}(t_2)\| < \epsilon$.*

3. *If a function $\vec{r}(t)$ is continuous on a closed and bounded set E, then $\vec{r}$ is uniformly continuous on E.*

Differentiability of Vector-Valued Functions

Definition (Differentiable)

A function $\vec{r}(t)$ is *differentiable* at $t_0 \in D$ iff the limit

$$\vec{r}\,'(t) = \lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}$$

exists and is finite.

Proposition

If f, g, and h are the components of $\vec{r}$, then $\vec{r}$ is differentiable iff f, g, and h are differentiable, whence

$$\vec{r}\,'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example

1. Find $\vec{r}\,'$ for the line through $P_0 = (1, 2, 4)$ parallel to $\vec{u} = (1, 2, -1)$.

Algebra of Vector-Valued Derivatives

Theorem (Algebra of Derivatives)

Suppose $\vec{u}, \vec{w} : D \to \mathbb{R}^n$ *&* $k : D \to \mathbb{R}$ *are all differentiable, and* $c \in \mathbb{R}$*. Then*

$$[\vec{u} \pm \vec{w}\,]' = [\vec{u}\,'] \pm [\vec{w}\,'] \tag{6}$$

$$[c\,\vec{u}\,]' = c\,[\vec{u}\,'] \tag{7}$$

$$[k\,\vec{u}\,]' = [k'] \; \vec{u} + k \; [\vec{u}\,'] \tag{8}$$

$$[\vec{u} \cdot \vec{w}\,]' = [\vec{u}\,'] \cdot \vec{w} + \vec{u} \cdot [\vec{w}\,'] \tag{9}$$

$$[\vec{u} \times \vec{w}\,]' = [\vec{u}\,'] \times \vec{w} + \vec{u} \times [\vec{w}\,'] \tag{10}$$

$$\|\vec{u}\|' = \frac{\vec{u} \cdot [\vec{u}\,']}{\|\vec{u}\|} \tag{11}$$

$$[\vec{u} \circ k]' = [\vec{u}\,' \circ k] * k' \tag{12}$$

Derivative Props

Properties

Suppose $\vec{r}(t)$ is a twice differentiable vector function.

1. $\vec{V}(t) = \vec{r}\,'(t)$ is
   - the *tangent vector* of $\vec{r}$
   - the *velocity vector* of $\vec{r}$

   and $S(t) = \|\vec{r}\,'(t)\|$ gives the *speed* of $\vec{r}(t)$
2. $\vec{A}(t) = \vec{V}\,'(t) = \vec{r}\,''(t)$ is
   - the *acceleration vector* of $\vec{r}$

Example

Find the velocity & acceleration and the speed for the function

1. $\vec{r}(t) = \langle 2\cos(t), 3\sin(t), z_0 \rangle$.
2. $\vec{\rho}(t) = \langle \cos(t)\cdot(1+\cos(t)), 2\sin(t)\cdot(1+t), t \rangle$.[1]

[1] `spacecurve(f(t),t=0..6*Pi,numpoints=101,thickness=3,axes=normal)`

Example 9.6.9

Example (9.6.9, pg 410)

Consider $\vec{u}, \vec{v}, \vec{w} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\vec{u} = \langle t, t^2 \rangle, \vec{v} = \langle t^3, t^6 \rangle, \text{ and } \vec{w} = \begin{cases} \langle t, t^2 \rangle & \text{if } t \leq 0 \\ \langle t^3, t^6 \rangle & \text{if } t > 0 \end{cases}$$

All 3 functions are continuous, all trace the parabola $y = x^2$, and all are $\vec{0}$ at $t = 0$.

1. $\vec{u}$ is differentiable at $t = 0$ with tangent vector $\vec{u}\,'(0) = \langle 1, 0 \rangle$ and tangent line $y = 0$.
2. $\vec{v}$ is differentiable at $t = 0$ with tangent vector $\vec{v}\,'(0) = \langle 0, 0 \rangle$, but has *no* tangent line $\vec{0}$.
3. $\vec{w}$ is *not* differentiable at $t = 0$ and has no tangent line at $\vec{0}$.

See Maple demo

Circles

Proposition

Let $\vec{r}$ be a differentiable vector function of t. Then $\|\vec{r}(t)\|$ is constant iff $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$; i.e. $\vec{r}$ and $\vec{r}\,'$ are orthogonal.

Proof.

$$\|\vec{r}(t)\| \text{ is constant } \iff \vec{r}(t) \cdot \vec{r}(t) = c \iff \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

□

Definition

Unit tangent vector: $\vec{T}(t) = \vec{r}\,'(t)/\|\vec{r}\,'(t)\|$

Unit normal vector: $\vec{N}(t) = \vec{T}'(t)/\|\vec{T}'(t)\|$

$\vec{V} = \vec{r}\,'$ and $v = \|\vec{V}\|$. Then $\vec{A} = \vec{V}' = v\,\vec{T}' + v'\,\vec{T}$. Since $\vec{T}' \perp \vec{T}$, then $\vec{A}_{\vec{N}} = v\,\vec{T}'$ and $\vec{A}_{\vec{T}} = v'\,\vec{T}$ forms an orthogonal decomp of $\vec{A}$

Þ^e Cræft

Project

Using

$$\vec{r}'' = \vec{A} = v\,\vec{T}' + v'\,\vec{T} \tag{13}$$

$$\vec{A} = \vec{A}_{\vec{N}} + \vec{A}_{\vec{T}} \tag{14}$$

1. *Compute* $\vec{A} \cdot \vec{T}$*?*
2. *What vector is* $(\vec{A} \cdot \vec{T})\,\vec{T}$*?*
3. *Compute* $\vec{A} - \left(\vec{A} \cdot \vec{T}\right)\vec{T}$*?*
4. *Apply this idea to* $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$*. What are* $\vec{A}$*'s orthognal components?*

Int

Definition

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

off the integrals exist. I.e., $\int_a^b \langle f_i\rangle(t)\,dt = \left\langle \int_a^b f_i(t)dt\right\rangle$.

Theorem (FToC)

Suppose $\vec{r}(t)$ is integrable on $[a,b]$ and $\vec{R}(t)$ is an antiderivative (or primitive) for $\vec{r}$. Then

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

Theorem

Suppose $\vec{r}(t)$ is integrable on $[a,b]$. Then

$$\left\|\int_a^b \vec{r}(t)\,dt\right\| \le \int_a^b \|\vec{r}(t)\|\ dt$$

Arclength

Definition (Arclength)

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is

$$L(\gamma) = \sup \{L_Q \mid Q \text{ partitions } [a, b]\}$$

where $L_Q = \sum_k \left\|\gamma(t_k) - \gamma(t_{k-1})\right\|$ for $t_k \in Q$.

Proposition

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is $L(\gamma) = \lim_{|Q|\to 0} L_Q$ where $|Q|$ is the norm of the partition.

Theorem (Useful Arclength Theorem)

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is

$$L(\gamma) = \int_a^b \sqrt{\sum_k (f_k')^2}\, dt = \int_a^b \left\|\vec{r}\,'(t)\right\|\, dt$$

Proof

Proof (UAT).

I. Let Q be a partition. Fix k. Whereupon

$$\sqrt{\sum_j [f_j(t_k)) - f_j(t_{k-1})]^2} = \|\vec{r}(t_k) - \vec{r}(t_{k-1})\| = \left\| \int_{t_{k-1}}^{t_k} \vec{r}\,'(t)\, dt \right\|$$

Since $\left\| \int \vec{r}\,' dt \right\| \le \int \|\vec{r}\,'\|\, dt$, then $L(\gamma) \le \int_a^b \|\vec{r}\,'(t)\|\, dt$.

II. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $\|\vec{r}(s) - \vec{r}(t)\| < \varepsilon$ for $|s - t| < \delta$. Choose $|Q| < \delta$.

1. $\displaystyle\int_{t_k}^{t_{k+1}} \|\vec{r}\,'(t)\|\, dt \le \int_{t_k}^{t_{k+1}} \|\vec{r}\,'(t_{k+1})\| + \varepsilon\, dt = \int_{t_k}^{t_{k+1}} \|\vec{r}\,'(t_{k+1})\|\, dt + \varepsilon\, \Delta t_k$
2. $\displaystyle\le \left\| \int_{t_k}^{t_{k+1}} \vec{r}\,'(t) dt \right\| + \left\| \int_{t_k}^{t_{k+1}} [\vec{r}\,'(t_{k+1}) - \vec{r}\,'(t)]\, dt \right\| + \varepsilon\, \Delta t_k$
3. $\displaystyle\le \|\vec{r}(t_{k+1}) - \vec{r}(t_k)\| + 2\varepsilon\, \Delta t_k \implies \int_a^b \|\vec{r}\,'(t)\|\, dt \le L_Q + 2\varepsilon(b - a)$

Hence $\int_a^b \|\vec{r}\,'(t)\|\, dt \le L(\gamma)$. □

Rectified

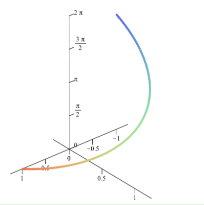
Definition (Recifiable Curve)

A curve γ is *rectifiable* iff $L(\gamma)$ is finite.

Examples (Curves[2])

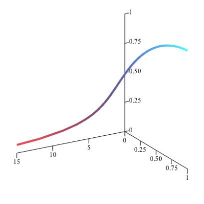
I. Let $\gamma(t) = \langle \cos(\pi t), \sin(\pi t), \sqrt{3}\,\pi t\rangle$ on $[0, 1]$.

1. $L(\gamma) = \int_0^1 \|\gamma'(t)\|\, dt$
2. $= \int_0^1 \left\|\pi\langle -\sin(\pi t), \cos(\pi t), \sqrt{3}\,\rangle\right\| dt = 2\pi$



II. Let $\psi(t) = \langle \tan(t), 1 - \sin(t), \cos(t)\rangle$ on $[0, \pi/2]$.

1. $L(\psi) = \int_0^1 \|\psi'(t)\|\, dt$
2. $= \int_0^1 \left\|\langle \sec^2(t), -\cos(t), -\sin(t)\rangle\right\| dt = \infty$



[2] Maple worksheet

Interlude

Theorem (Most Useful Norm-Integral Estimate)

Let $\vec{r}(t)$ be Riemann integrable on $[a, b]$. Then $\|\vec{r}(t)\|$ is integrable and

$$\left\| \int_a^b \vec{r}(t)\, dt \right\| \leq \int_a^b \|\vec{r}(t)\|\; dt$$

Proof.

I. $\|\vec{r}(t)\|$ is integrable: ✓

II. $(\mathbb{R}^2)$.
$$\left\| \int_a^b \vec{r}(t)\, dt \right\| = \sqrt{\left(\int_a^b f\right)^2 + \left(\int_a^b g\right)^2}$$
$$\leq \sqrt{\int_a^b (f^2) + \int_a^b (g^2)} = \sqrt{\int_a^b (f^2 + g^2)}$$
$$\leq \int_a^b \sqrt{f^2 + g^2} = \int_a^b \|\vec{r}(t)\|\; dt.$$
□

Reparametrize

Definition

Two parametrizations γ_1 on $[a, b]$ and γ_2 on $[c, d]$ of a curve are *equivalent* iff there is a continuously differentiable bijection $u : [c, d] \to [a, b]$ such that $u(c) = a$, $u(d) = b$, and $\gamma_2 = \gamma_1 \circ u$.

Theorem

Suppose γ_1 and γ_2 are equivalent smooth parametrizations of a curve. Then $L(\gamma_1) = L(\gamma_2)$.

Proof.

Let u be the equivalence bijection for γ_1 and γ_2. Then

$$L(\gamma_2) = \int_c^d \|\gamma_2'(t)\| dt = \int_c^d \|\gamma_1'(u(t)) \cdot u'(t)\| dt$$

$$= \int_c^d \|\gamma_1'(u(t))\| \cdot u'(t)\, dt = \int_a^b \|\gamma_1(s)\| ds = L(\gamma_1)$$

□

Parametrization by Arclength

Definition (Arclength Parameter)

Set $\ell(t) = \int_a^t \|\vec{r}\,'(t)\| dt$. Then ℓ is continuous, differentiable, a bijection, and increasing $\Rightarrow$ it has an inverse $\ell^{-1} : [0, L(\gamma)] \to [a, b]$. So $\gamma \circ \ell^{-1} : [0, L(\gamma)] \to \mathbb{R}^n$ is the *arclength parametrization* of γ.

Example

Let $\vec{r}(t) = \langle \cos(t), \sin(t), t/3 \rangle$ on $[-4\pi, 4\pi]$.

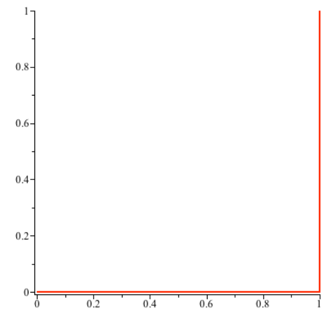
1. Whence $\|\vec{r}\,'(t)\| = \|\langle -\sin(t), \cos(t), 1/3 \rangle\| = \sqrt{10}/3$.
2. Hence $\ell(t) = \int_{-4\pi}^t \sqrt{10}/3\, dt = \sqrt{10}/3 \cdot (t + 4\pi)$.
3. Fortuitously, ℓ is algebraically invertible (*usually not true!*) and $\ell^{-1}(s) = (3/\sqrt{10})s - 4\pi$.
4. Whereupon the arc length parametrized form of γ is

$$\gamma(s) = \left\langle \cos\left(\frac{3}{\sqrt{10}}\, s\right), \sin\left(\frac{3}{\sqrt{10}}\, s\right), \frac{1}{\sqrt{10}}\, s - \frac{4}{3}\,\pi \right\rangle \quad \text{on } \left[0, \frac{8\sqrt{10}}{3}\,\pi\right]$$

Maple

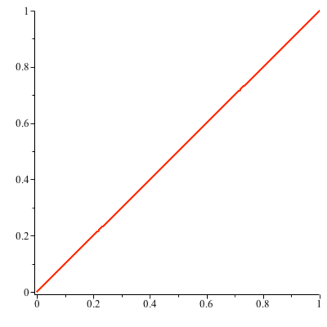
What's the Problem?

Example $(2 \to \sqrt{2})$



$L(\gamma_n) = 2$

$$\xrightarrow[n\to\infty]{}$$



$L(\gamma_\infty) = \sqrt{2}$

Maple

Interlude: Inner Products

Definition (Inner Product)

Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space V over the field F, and that $c \in F$ is a scalar. An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ such that

1. $\langle \vec{u}, \vec{w} \rangle = \langle \vec{w}, \vec{u} \rangle$ — *commutivity*
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ — *additivity* } *bi-linearity*
3. $\langle c\,\vec{u}, \vec{v} \rangle = c\,\langle \vec{u}, \vec{v} \rangle$ — *scalar homogeneity* } *bi-linearity*
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$ } *positive definite*
5. $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$ } *positive definite*

Examples

1. The usual dot product on $\mathbb{R}^3$.
2. For $p(x) = \sum^{n} a_j x^j$, $q(x) = \sum^{n} b_j x^j \in \mathbb{P}^n$, set $\langle p, q \rangle = \sum^{n} a_i b_i$.

Interlude: Orthogonality

Proposition

Suppose that $f(x), g(x) : [a, b] \to \mathbb{R}$ are (piecewise) continuous functions. Then

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

is an inner product on the vector space of (piecewise) continuous functions on $[a, b]$

Definition (Orthogonal Vectors)

Suppose that $\vec{u}$ and $\vec{w}$ are vectors in a vector space V over the field F. Then $\vec{u}$ is orthogonal to $\vec{w}$ iff $\langle \vec{u}, \vec{w} \rangle = 0$.

Example (Orthogonal Functions)

1. $\langle \sin, \cos \rangle = \int_{-\pi}^{\pi} \sin(\theta)\cos(\theta)\,d\theta = 0 \implies$ sine $\perp$ cosine on $[-\pi, \pi]$

Interlude: Orthogonal Polynomials

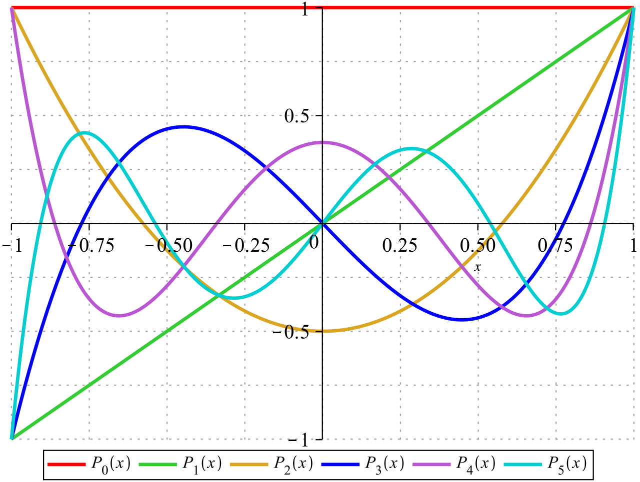
Example (The Legendre Polynomials)

The Legendre polynomials are orthogonal on $[-1,1]$ wrt $\langle f,g\rangle = \int_{-1}^{1} f g\,dx$. Two formulas for the Legendre polynomials P_n are

1. Rodrigues' formula: $\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$.
2. recurrence relation: $(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - nP_{n-1}(x)$.

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \tfrac{1}{2}\left(3x^2-1\right)\\
P_3(x) &= \tfrac{1}{2}\left(5x^3-3x\right)\\
P_4(x) &= \tfrac{1}{8}\left(35x^4-30x^2+3\right)\\
P_5(x) &= \tfrac{1}{8}\left(63x^5-70x^3+15x\right)\\
P_6(x) &= \tfrac{1}{16}\left(231x^6-315x^4+105x^2-5\right)\\
P_7(x) &= \tfrac{1}{16}\left(428x^7-693x^5+315x^3-35x\right)
\end{aligned}
$$

Interlude: Legendre Polynomials' Graphs



Maple

Interlude: Expansions in Legendre Polynomials

Proposition (Orthonormalized Legendre Polynomials)

Let $p_n(x) = \sqrt{\frac{2n+1}{2}} \cdot P_n(x)$*. Then* $\langle p_n, p_m \rangle = \delta_{m,n}$*.*

Theorem

Let f *be integrable on* $[-1, 1]$*, and set* $a_n = \langle f, p_n \rangle$*. Then*

$$f_n(x) = \sum_{k=0}^{n} a_n \, p_n(x) \xrightarrow[n]{} f(x)$$

Example

For $f(x) = \sin(\pi x)$ on $[0, a]$, we have

$$a := \left[0, \frac{\sqrt{6}}{\pi}, 0, \frac{\sqrt{14}}{\pi^3}\left(\pi^2 - 15\right), 0, \frac{\sqrt{22}}{\pi^5}\left(\pi^4 - 105\pi^2 + 945\right), 0, \dots\right]$$

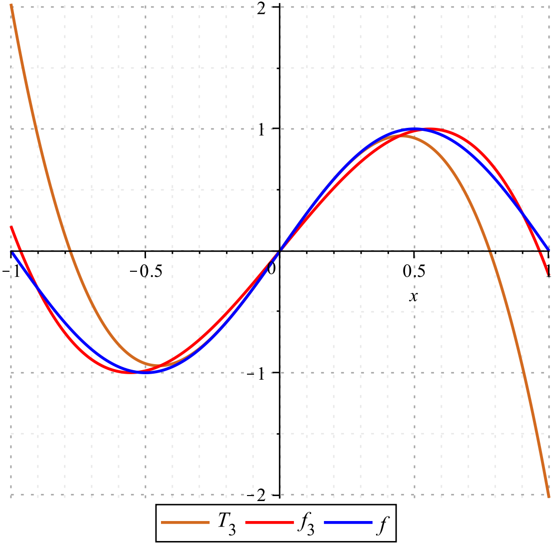
$$\sin_3(x) = \frac{\sqrt{6}}{\pi} p_1(x) + \frac{\sqrt{14}}{\pi^3}\left(\pi^2 - 15\right) p_3(x) = -\frac{15}{2}\frac{\pi^2 - 21}{\pi^3} x + \frac{35}{2}\frac{\pi^2 - 15}{\pi^3} x^3$$

Interlude: Legendre Expansion Graph

$f(x) = \sin(\pi x)$

$f_3(x)$: Legendre expansion

$T_3(x)$: Taylor expansion



Basic Topology of $\mathbb{R}^n$

Definition (*Total Recall:)*)

Open ball:	$B(\vec{c}; r) = \{\vec{x} \mid \|\vec{x} - \vec{c}\| < r\} \subseteq \mathbb{R}^n$
Punct'd ball:	$B'(\vec{c}; r) = \{\vec{x} \mid 0 < \|\vec{x} - \vec{c}\| < r\} \subset \mathbb{R}^n$; NB: $\vec{c} \notin B'(\vec{c}; r)$
Interior point:	$\vec{a} \in \operatorname{int}(S)$ iff $\exists\, \varepsilon > 0$ such that $B(\vec{a}; \varepsilon) \subset S$
Open set:	S is *open* iff $S = \operatorname{int}(S)$
Accum point:	$\vec{a}$ in an *accumulation pt* of S iff $\forall \varepsilon > 0\ \ [B'(\vec{a}; \varepsilon) \cap S] \neq \emptyset$
Derived set:	$S' = \{$all accumulation pts of $S\}$
Closed set:	S is *closed* iff $S' \subseteq S$
Closure:	The closure of S is $\overline{S} = S \cup S'$
Boundary pt:	$\vec{b}$ is a *boundary pt* of S iff $B(\vec{b}; \varepsilon)$ contains points both of S and S complement for all $\varepsilon > 0$
Boundary:	$\partial S = \{$all boundary pts of $S\}$
Isolated pt:	$\vec{a}$ in an *isolated pt* of S iff $\exists\, \varepsilon > 0\ \ [B'(\vec{a}; \varepsilon) \cap S] = \emptyset$

Proper Stichens

Proposition (Open Sets)

1. *If $\mathcal{I}$ is an indexing set for a family of open sets $\{O_i\}$, then the set $\mathcal{O} = \bigcup_{i\in\mathcal{I}} O_i$ is open.* *(Arbitrary unions of open sets are open.)*
2. *If $\{O_i\}_{i=1}^{n}$ is a finite family of open sets, then $\mathcal{O} = \bigcap_{i=1}^{n} O_i$ is open.* *(Finite intersections of open sets are open.)*

Examples

1. Let $O_x = (-x, x)$ for $x \in (0,1) = \mathcal{I}$. Then
$$\bigcup_{i\in\mathcal{I}} O_i = ? \qquad \bigcap_{i\in\mathcal{I}} O_i = ?$$
2. Let $P_i = \left(-1 - \frac{1}{i}, 1 - \frac{1}{i}\right)$ for $i = 1..n$. Then
$$\bigcap_{i=1}^{n} P_i = ? \qquad \bigcup_{i=1}^{n} P_i = ?$$

Closed to Stichens

Proposition (Closed Sets)

1. *If $\mathcal{I}$ is an indexing set for a family of closed sets $\{F_i\}$, then the set $\mathcal{F} = \bigcap_{i \in \mathcal{I}} F_i$ is closed.* *(Arbitrary intersections of closed sets are closed.)*

2. *If $\{F_i\}_{i=1}^{n}$ is a finite family of closed sets, then $\mathcal{O} = \bigcup_{i=1}^{n} F_i$ is closed.* *(Finite unions of closed sets are closed.)*

Examples

1. Let $F_k = \left[-1 + \frac{1}{k}, 1 - \frac{1}{k}\right]$ for $k \in \mathbb{N}$. Then

$$\bigcap_{k \in \mathbb{N}} F_k = ? \qquad \bigcup_{k \in \mathbb{N}} F_k = ?$$

2. Let $H_i = \left[-1, 1 - \frac{1}{i}\right]$ for $i = 1..n$. Then

$$\bigcap_{i=1}^{n} H_i = ? \qquad \bigcup_{i=1}^{n} H_i = ?$$

Proper Themes

Theorem (Bolzano-Weierstrass Theorem)

A bounded, infinite subset of $\mathbb{R}^n$ *has an accumulation point.*

Proof.

Lion in the desert. □

Theorem (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ *is compact iff it is closed and bounded.*

Theorem (Cantor Intersection Theorem)

Let $\{F_k\}$ *be a sequence of nested* $(F_{k+1} \subseteq F_k)$*, closed, nonempty sets for* $k \in \mathbb{N}$ *with* F_1 *being bounded. Then*

$$F = \bigcap_{k=1}^{n} F_k$$

is closed and nonempty.

CIT

Proof. (Cantor Intersection Theorem).

I. If F is finite for some, then done.

II. Each F_n is infinite. Define $S = \bigcap_{k=1}^{\infty} F_k$.

1. S is closed.
2. 2.a Define the sequence $A = \{a_k\}$ by choosing distinct points $a_k \in F_k$ for each k. *Uses: F_k's are infinite.*

   2.b Since F_1 is bounded, the sequence forms a bounded, infinite set.

   2.c Therefore A has an accumulation pt a. *Bolzano-Weierstrass!*

   2.d Let $r > 0$ and set $B = B'(a; r)$. Since a is an acc pt of A, then B contains ∞ many pts of A. As the F_k's are nested, B also must contain ∞ many pts of F_k. Whence a is an acc pt of F_k.

   2.e F_k is closed, so $a \in F_k$.

   2.f The F_k are nested, so $a \in \bigcap_k F_k$; i.e., the intersection is nonempty.

□

Sample Intersections

Examples (CIT)

1. Define: $F_0 = [0,1]$; $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = F_0 - (\frac{1}{3}, \frac{2}{3})$; $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$; &c. Hence
$$F_n = \bigcup_{k=0}^{\lfloor 3^n/2 \rfloor} \left[\frac{2k}{3^n}, \frac{2k+1}{3^n}\right]_{J(k,n)}$$
Let $\mathcal{C} = \bigcap_n F_n$. Whence *CIT* $\implies$ $\mathcal{C}$ is nonempty and closed.

2. Let $H_n = [n, \infty)$. Then H_n is a sequence of nested, closed sets. But $\bigcap_n H_n = ?$

3. Set $J_n = (-\frac{n+1}{n^2}, \frac{n+1}{n^2})$. Then J_n is a sequence of bounded, nested sets.
But $\bigcap_n J_n = ?$

Disconnection

Connected and Separated Sets

Separated: Two sets A and B are *separated* iff $A \cap \overline{B} = \emptyset = \overline{A} \cap B$.

Connected: A set S is *connected* iff S is not the union of 2 nonempty, separated sets.

Arcwise conn: Any two points in S are conn by a path inside S.

Disconnected: A set is *disconnected* iff S is not connected.

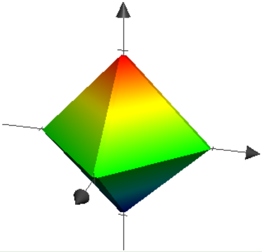
Region: A *region* is a connected set that may contain boundary points (may be neither open or closed).

Proposition

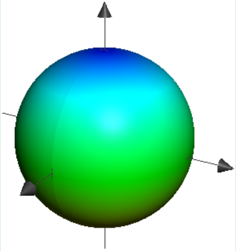
1. *Disjoint sets are separated if neither contains acc pts of the other.*
2. *Arcwise connected sets are connected*
3. *A nonempty, open, connected set is arcwise connected.*

Interlude

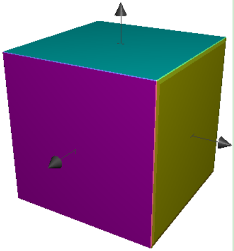
Example (Unit Balls in $\mathbb{R}^2$)



$|x| + |y| = 1$



$\sqrt{x^2 + y^2} = 1$



$\max(|x|, |y|) = 1$

Proposition

The open sets are the same under each of the metrics above.

Limits and Continuity

Definition (Limit)

- Let $f : D \to \mathbb{R}$, and let $(a, b) \in D' \subseteq \mathbb{R}^2$. Then
$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$
iff $[\forall\varepsilon > 0]\ [\exists\delta > 0]\ [\forall(x, y) \in D]$, if $\|(x, y) - (a, b)\| < \delta$, then $|f(x, y) - L)| < \varepsilon$.

- Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D' \subseteq \mathbb{R}^n$. Then
$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$
iff $[\forall\varepsilon > 0]\ [\exists\delta > 0]\ [\forall\vec{x} \in D]$, if $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - L| < \varepsilon$.

- Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D' \subseteq \mathbb{R}^n$. Then
$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$
iff $[\forall\varepsilon > 0]\ [\exists\delta > 0]$, $f\big(D \cap B'(\vec{a}; \delta)\big) \subseteq B(L; \varepsilon)$.

Limiting Examples

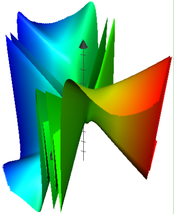
Example (*Good Function! Biscuit!*)

Let $f(x, y) = x\sin(1/y) + y\sin(1/x)$. Then

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

Proof. Let $\delta(\varepsilon) = \varepsilon/2$. And

$$|f(x, y)| \leq |x| + |y|$$

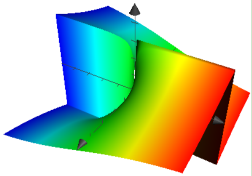


Example (*Bad Function! No biscuit!*)

Let $g(x, y) = \arctan(y/x)$. Then

$$\lim_{(x,y)\to(0,0)} g(x, y) \text{ \textit{D.N.E.}}$$

Proof. Observe that $\lim_{t\to 0} g(t, t) = \pi/4$ and $\lim_{t\to 0} g(-t, t) = -\pi/4$.



Algebra of Limits

Theorem (The Algebra of Limits)

Let $f, g : D \to \mathbb{R}$ *and* $\vec{a} \in D'$. *Suppose* $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L_f$ *and* $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = L_g$. *Then*

1. $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = L_f \pm L_g$
2. $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) \cdot g(\vec{x}) = L_f \cdot L_g$
3. $\lim\limits_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{L_f}{L_g}$ *as long as* $L_g \neq 0$
4. $\lim\limits_{\vec{x}\to\vec{a}} |f(\vec{x})| = |L_f|$
5. *if* $f(\vec{x}) \underset{(\leq)}{<} g(\vec{x})$ *on some* $B'(\vec{a}; r)$, *then* $L_f \leq L_g$

Continuity

Definition (Continuity)

Let $f : D \to \mathbb{R}$, and $(a, b) \in D \subseteq \mathbb{R}^2$. Then f is *continuous* at (a, b) iff

- $[\forall \varepsilon > 0]\ [\exists \delta > 0]\ [\forall (x, y) \in D]$, if $\|(x, y) - (a, b)\| < \delta$, then $|f(x, y) - f(a, b))| < \varepsilon$.

Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D \subseteq \mathbb{R}^n$. Then f is *continuous* at $\vec{a}$ iff

- $[\forall \varepsilon > 0]\ [\exists \delta > 0]\ [\forall \vec{x} \in D]$, if $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

- $[\forall \varepsilon > 0]\ [\exists \delta > 0]\ \ f\big(D \cap B(\vec{a}; \delta)\big) \subseteq B\big(f(\vec{a}); \varepsilon\big)$.

- $[\forall O \subseteq \mathbb{R}$, open set$]\ \ f^{-1}(O) \subseteq \mathbb{R}^n$ is an open set.

Proposition

f is continuous at $\vec{a}$ iff $\big[\forall \{\vec{a}_n\}\big]$ if $\vec{a}_n \to \vec{a}$, then $f(\vec{a}_n) \to f(\vec{a})$

Algebra of Continuity

Theorem (The Algebra of Continuity)

Let $f, g : D \to \mathbb{R}$ be continuous at $\vec{a} \in D$. Then

1. $f \pm g$ *is continuous at* $\vec{a}$
2. $f \cdot g$ *is continuous at* $\vec{a}$
3. f/g *is continuous at* $\vec{a}$ *as long as* $g(\vec{a}) \neq 0$
4. $|f|$ *is continuous at* $\vec{a}$

Proof.

2. $(D \subseteq \mathbb{R}^2)$ Let $\vec{a}_n \to \vec{a}$. Since $(fg)(\vec{a}_n) = f(\vec{a}_n)\, g(\vec{a}_n)$, and f & g are continuous at $\vec{a}$, we have $f(\vec{a}_n)\, g(\vec{a}_n) \to f(\vec{a})\, g(\vec{a}) = (fg)(\vec{a})$. Thus $(fg)(\vec{a}_n) \to (fg)(\vec{a})$ for any sequence $\vec{a}_n \to \vec{a}$; hence, fg is continuous at $\vec{a}$. □

(Note: Thm 10.2.9 has problems: g & f can't be composed as $\mathrm{range}(f) \subset \mathbb{R}^1$, but $\mathrm{dom}(g) \subset \mathbb{R}^2$. So $\mathrm{range}(f) \not\subseteq \mathrm{dom}(g)$.

Continuously Reverted

Proposition

$f : \mathbb{R}^n \to \mathbb{R}$ is continuous iff

- *the preimage of any open set (in $\mathbb{R}^1$) is open (in $\mathbb{R}^n$).*
- *the preimage of any closed set (in $\mathbb{R}^1$) is closed (in $\mathbb{R}^n$).*

Proof.

($\Rightarrow$) Assume f is cont and S is open in $\mathbb{R}^1$.
Let $\vec{a} \in f^{-1}(S)$; i.e. $f(\vec{a}) \in S$. For some $r > 0$, then $B(f(a); r) \subseteq S$.
Whence there is a $\delta > 0$, s.t. $f(B(\vec{a}; \delta)) \subseteq B(f(a); r) \subseteq S$.
Hence $B(\vec{a}; \delta) \subseteq f^{-1}(S)$.

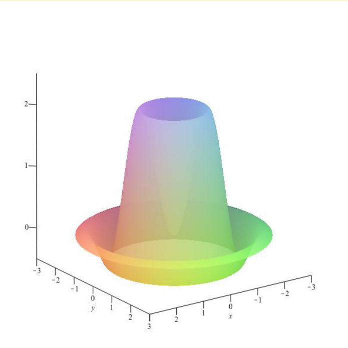
($\Leftarrow$) Assume $f^{-1}(S)$ is open whenever S is open.
Let $\vec{a} \in f^{-1}(S)$ and $\varepsilon > 0$. Thence $f^{-1}(B(f(\vec{a}; \varepsilon))$ is open.
Thus there is a $\delta > 0$ s.t. $B(\vec{a}; \delta) \subseteq f^{-1}(B(f(\vec{a}; \varepsilon))$.
Apply f to have $f(B(\vec{a}; \delta)) \subseteq B(f(\vec{a}; \varepsilon))$.

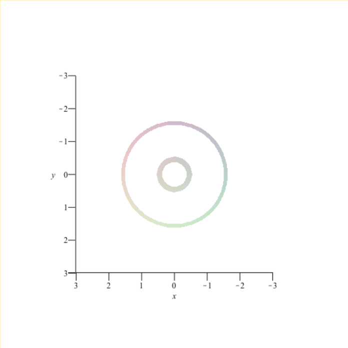
□

Continuously Pictured

Preimage

Let $f(x,y) = 4\,\sin(x^2+y^2)\,e^{(-(x^2+y^2)/2)}$





$$S = \left(\frac{1}{2}, 1\right) \implies f^{-1}(S) = \{0.37 < \|\vec{x}\| < 0.54\} \bigcup \{1.50 < \|\vec{x})\| < 1.78\}$$

Uniform

Definition (Uniform Continuity)

A function $f : D \to \mathbb{R}$ is *uniformly continuous on* D iff for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $\vec{x}_1, \vec{x}_2 \in D$, if $\|\vec{x}_1 - \vec{x}_2\| < \delta$, then $|f(\vec{x}_1) - f(\vec{x}_2)| < \varepsilon$.

Theorem

If f is continuous on D, and D is closed & bounded (compact), then

1. *f is bounded,*
2. *f attains extreme values (max and min),*
3. *f is uniformly continuous on D.*

Proof (Homework).

1. Hint: Assume not, then look at $f^{-1}(a_n)$ where $a_n \to \infty$.
2. *Bolzano-Weierstrass in action.*
3. Hint: Assume not. Create sequences $\vec{x}_n$, $\vec{y}_n$ that converge to $\vec{a}$, but have $|f(\vec{x}_n) - f(\vec{y}_n)| > \varepsilon$. Cont gives a contradiction. □

Connecting to Rudolph Otto

Theorem

Let $f\colon D \to \mathbb{R}$ be continuous and let S be a connected subset of D. Then $f(S)$ is connected. (A connected set in $\mathbb{R}$ is an interval.)

Proof.

Suppose $f(S) = A \cup B$ with A & B nonempty, separated sets in $\mathbb{R}$.
Define $G = S \cap f^{-1}(A)$ and $H = S \cap f^{-1}(B)$.

1. $S = G \cup H$ since $f\colon S \underset{\text{onto}}{\longrightarrow} f(S)$.
2. Let $\vec{y} \in A$. ($A \neq \emptyset$.) $\exists \vec{x} \in S$ s.t. $f(\vec{x}) = \vec{y}$. Thus $\vec{x} \in G \implies G \neq \emptyset$. Similarly, $H \neq \emptyset$.
3. Let $\vec{p} \in \overline{G} \cap H$. If $\vec{p} \in G$, then $\vec{p} \in G \cap H$. Then $\vec{p} \in f^{-1}(A \cap B)$; i.e., $f(\vec{p}) \in A \cap B = \emptyset$. Thus $\vec{p} \notin G$, whence $\vec{p} \in G'$ and $f(\vec{p}) \in B$. Since $\overline{A} \cap B = \emptyset$ and $\vec{p} \in B$, $\exists \varepsilon > 0$ s.t. $B(f(\vec{p}); \varepsilon) \cap A = \emptyset$. Since f is cont, $\exists \delta > 0$ s.t. $f(B(\vec{p}; \delta)) \subset B(f(\vec{p}); \varepsilon)$. Then $B(\vec{p}; \delta) \cap G$ is empty contrary to $\vec{p} \in G'$. Hence $\overline{G} \cap H = \emptyset$. Similarly $G \cap \overline{H} = \emptyset$.
4. Whereupon S is separated by G and H. *oops* $\rightarrow\leftarrow$ □

Fun with Functions

Problem (Functions)

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a function. Let A and B be subsets of the domain and range of f, respectively. Then

$$f(A) = \{y \in \mathbb{R} \mid f(a) = y \text{ for some } a \in A\} \subseteq \operatorname{range}(f)$$
$$f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) = b \text{ for some } b \in B\} \subseteq \operatorname{dom}(f)$$

Give an example justifying your answer.

1. **T** or **F**: $A \subseteq f^{-1}\big(f(A)\big)$
2. **T** or **F**: $A = f^{-1}\big(f(A)\big)$
3. **T** or **F**: $A \supseteq f^{-1}\big(f(A)\big)$ *or* $f^{-1}\big(f(A)\big) \subseteq A$
4. **T** or **F**: $B \subseteq f\big(f^{-1}(B)\big)$
5. **T** or **F**: $B = f\big(f^{-1}(B)\big)$
6. **T** or **F**: $B \supseteq f\big(f^{-1}(B)\big)$ *or* $f\big(f^{-1}(B)\big) \subseteq B$

Rudolph Otto S von L

Definition (Lipschitz Condition)

If there is a constant L s.t.

$$|f(\vec{x}_1) - f(\vec{x}_2)| \leq L \, \|\vec{x}_1 - \vec{x}_1\|$$

for all $f\vec{x}_1, \vec{x}_2 \in D$, then f satisfies a *Lipschitz condition on D* (also called a "Lipschitz 1" condition).

Proposition

A function that is Lipschitz on D is uniformly continuous on D.

Proof.

Suppose f is Lipschitz with constant L.
Let $\varepsilon > 0$. Choose $0 < \delta < \varepsilon/L$. For any vectors $\vec{x}_1$ and $\vec{x}_2$ in $\mathrm{dom}(f)$ with $\|\vec{x}_1 - \vec{x}_2\| < \delta$, we have

$$|f(\vec{x}_1) - f(\vec{x}_2)| \leq L\|\vec{x}_1 - \vec{x}_2\| < L\delta < \varepsilon$$

□

Exercise

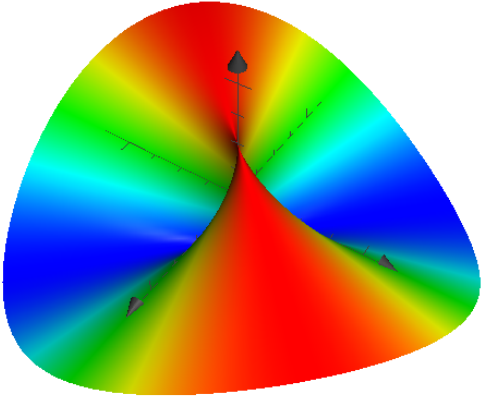
Problem (#14, pg 447)

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(r,\theta) = \begin{cases} \frac{1}{2}\sin(2\theta) & r \neq 0 \\ 0 & r = 0 \end{cases}$$

1. *Is* f *continuous in polar coordinates?*
   Let $\theta = \pm\pi/4$*, resp., and* $r \to 0$*. Then* $\lim_{(r,\pi/4)\to\vec{0}} f(r,\theta) = 1/2$*, but* $\lim_{(r,-\pi/4)\to\vec{0}} f(r,\theta) = -1/2$*. Thus,* f *is not continuous at* $\vec{0}$ *(polar).*
2. *Write* f *in rectangular coordinates.*
   $$\tfrac{1}{2}\sin(2\theta) = \cos(\theta)\sin(\theta) = \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2}$$
3. *Is* f *in rectangular coordinates continuous?*
   Let $(x,y) \to \vec{0}$ *as* (t,t) *and as* $(t,-t)$*. Then* $f \to \pm 1/2$ *as* $t \to 0$*. Hence* f *is not continuous at* $\vec{0}$*.*

Exercise's Graph



$$f(r,\theta) = \begin{cases} \frac{1}{2}\sin(2\theta) & r \neq 0 \\ 0 & r = 0 \end{cases} \iff f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

.

Challenge Problem

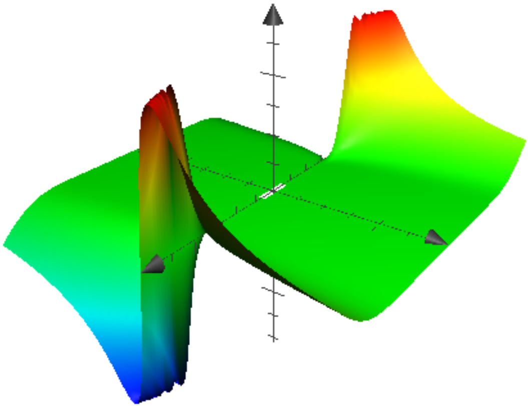
Problem (*Hmm.*)

Define $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$\varphi(x,y) = \begin{cases} \dfrac{e^{-1/x^2} y}{e^{-2/x^2} + y^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

1. *Let* C *be an arbitrary curve* $y = c\,x^{m/n}$ *for* $m, n \in \mathbb{N}$ *with* n*: odd. Find*
$$\lim_{x \to 0} \varphi(x, c\,x^{m/n})$$
2. *Define the sequence* $\vec{a}_n = \left(\frac{1}{n}, e^{-n^2}\right)$*. Find*
$$\lim_{n \to \infty} \vec{a_n} \qquad \textit{and} \qquad \lim_{n \to \infty} \varphi(\vec{a}_n)$$
3. *Is* φ *continuous at* $\vec{0}$*?*

The Challenge Problem Plot Thickens



$$\varphi(x,y) = \begin{cases} \dfrac{e^{-1/x^2} y}{e^{-2/x^2} + y^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Partial Derivatives

Definition (Partial Derivatives)

Let D be an open set in $\mathbb{R}^2$, $(a,b) \in D$, and $f : D \to \mathbb{R}$. Then

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

when the limits are finite.

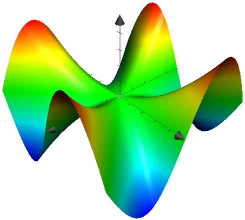
Example (*Woof!*)

Let $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ and $f(\vec{0}) = 0$. Then

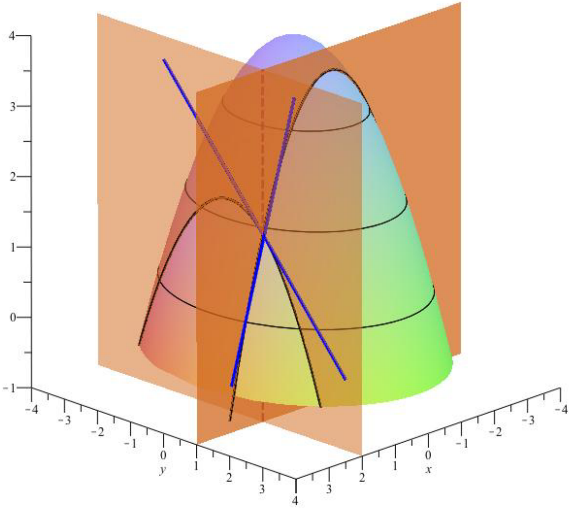
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-0}{h} = 0$$



Picture Time



$$f(x, y) = 4 - \frac{1}{2}\,x^2 - \frac{1}{3}\,y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(2,1)\ \&\ \frac{\partial f}{\partial x}(2,1)$$

▸ Differentiation

More Partial Derivatives

Examples

1. $h(x,y) = x^2/\sqrt{y}$. Then
$$h_x(x,y) = 2x\,y^{-1/2}$$
$$h_y(x,y) = -\tfrac{1}{2}x^2y^{-3/2}$$

2. $g(x,y) = -\cos(x+y^2)$. Then
$$g_x(x,y) = \sin(x+y^2)$$
$$g_y(x,y) = 2y\sin(x+y^2)$$

3. $f(x,y) = x^2\sin(y) - xe^{-xy}$. Then
$$f_x(x,y) = 2x\sin(y) + (xy-1)e^{-xy}$$
$$f_y(x,y) = x^2\left(\cos(y) + e^{-xy}\right)$$

Deeper Partial Derivatives

Theorem (Clairaut's[3] Theorem (1743))

Let $D \subset \mathbb{R}^2$ be open and $f : D \to \mathbb{R}$. If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous on D, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on D.

Proof.

Let $(a, b) \in D$. Set

$$g(h, k) = f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b)$$
$$p(x, y) = f(x + h, y) - f(x, y) = \Delta_x f$$
$$q(x, y) = f(x, y + k) - f(x, y) = \Delta_y f$$

Then

$$g(h, k) = p(a, b + k) - p(a, b) = \Delta_y p = \Delta_y \Delta_x f$$
$$g(h, k) = q(a + h, b) - q(a, b) = \Delta_x q = \Delta_x \Delta_y f$$

[3] *Presented his first paper at age 13; only one of his 19 siblings to reach adulthood.*

Deeper Partial Derivatives, II

Proof (cont).

Apply the MVT to $\Delta_y p$ and $\Delta_x q$ above to have (for some $\theta_j \in (0,1)$)

$$g(h,k) = k\, p_y(a, b+\theta_1 k) = k \cdot [f_y(a+h, b+\theta_1 k) - f_y(a, b+\theta_1 k)]$$

$$g(h,k) = h\, q_x(a+\theta_2 h, b)) = h \cdot [f_x(a+\theta_2 h, b+k) - f_x(a+\theta_2 h, b)]$$

Apply the MVT to $\Delta_x f_y$ and $\Delta_y f_x$ above to have (for some $\theta_k \in (0,1)$).

$$g(h,k) = hk\, f_{yx}(a+\theta_3 h, b+\theta_1 k)$$
$$g(h,k) = kh\, f_{xy}(a+\theta_2 h, b+\theta_4 k)$$

Whence

$$f_{yx}(a+\theta_3 h, b+\theta_1 k) = f_{xy}(a+\theta_2 h, b+\theta_4 k)$$

Let $h, k \to 0$. Since f_{xy} and f_{yx} are continuous, then

$$f_{yx}(a,b) = f_{xy}(a,b)$$

□

Deeper Samples

Examples

1. $g(x,y) = -\cos(x+y^2)$. Then

$$g_x(x,y) = \sin(x+y^2) \implies g_{xy}(x,y) = 2y\cos(x+y^2)$$

$$g_y(x,y) = 2y\sin(x+y^2) \implies g_{yx}(x,y) = 2y\cos(x+y^2)$$

2. $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$. Then (Maple)

$$f_y(x,0) = \begin{cases} x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f_x(0,y) = \begin{cases} -y & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Whence $f_{xy}(0,0) = -1$, but $f_{yx}(0,0) = +1$.

Operators and Exact Equations

Definition (Operators and Annihilators)

Let $C^1(S) = \{$*continuously differentiable fcns on* $S\}$.

- An *operator* on S is a fcn $\Phi : C^1(S) \to C^1(S)$.
- An *annihilator* is an operator combination that maps a fcn to 0.

Definition (Exact Differential Equations)

A differential equation $M\,dx + N\,dy = 0$ is *exact* iff there is a function $f(x, y)$ s.t. $M = \partial f/\partial x$ and $N = \partial f/\partial y$.

Examples

- $D_j = \frac{\partial}{\partial x_j}$ is an operator on $C^1(\mathbb{R}^n)$.
- $L = (D-2)^2$ annihilates the function $f_a(x) = axe^{2x}$.
- The DE $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$ is exact from $f(x, y) = x^2y + x\,y^2$.

Partial Antiderivatives and Exact Equations

Example

Solve the DE: $2xy\,dx + (x^2-1)\,dy = 0$

Solution: Set $M = 2xy$ and $N = x^2 - 1$.

1. Since $f_x = M = 2xy$, then $f(x,y) = \int 2xy\,dx = x^2y + \phi(y)$.
   partial antiderivative
2. Now $f_y = N = (x^2-1)$, so
   $$\frac{\partial}{\partial y}\left[x^2y + \phi(y)\right] = x^2 - 1.$$
   Since $\frac{\partial}{\partial y}\left[x^2y + \phi(y)\right] = x^2 + \frac{d}{dy}\phi(y)$, we have $\phi'(y) = -1$.
   Whence $\phi(y) = -y$

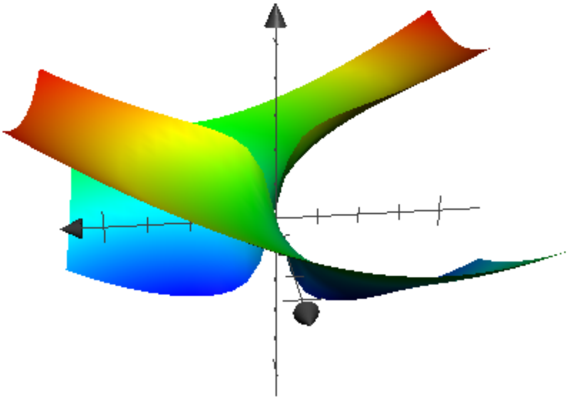
Putting the pieces together, $f(x,y)$ is given by

$$x^2y - y = c$$

where c is a constant of integration.

Try: $\big(x + y/(x^2+y^2)\big)\,dx + \big(y - x/(x^2+y^2)\big)\,dy = 0.$

Picture Time Again



$$f(x,y) = \frac{1}{2}(x^2 + y^2) + \arctan\left(\frac{x}{y}\right)$$

Tangent Plane

Consider...

In $\mathbb{R}^2$

- Slope of the tangent line at $x = a$ is $f'(a)$
- Tangent line is $y = f(a) + f'(a)(x - a)$

In $\mathbb{R}^3$

- Tangent vector in the x direction at $\vec{a}$ is $T_x = \langle 1, 0, f_x(\vec{a}) \rangle$
- Tangent vector in the y direction at $\vec{a}$ is $T_y = \langle 0, 1, f_y(\vec{a}) \rangle$
- A plane containing $\vec{a}$ and the tangent vectors is

$$(T_x \times T_y) \cdot (\vec{x} - \vec{a}) = 0$$

or (with $\vec{a} = \langle x_0, y_0 \rangle$ and $\vec{m}_{\vec{a}} = \langle f_x(\vec{a}), f_y(\vec{a}) \rangle$)

$$\begin{aligned} z &= f(\vec{a}) + f_x(\vec{a})(x - x_0) + f_y(\vec{a})(y - y_0) \\ &= f(\vec{a}) + \vec{m}_{\vec{a}} \cdot (\vec{x} - \vec{a}) \end{aligned}$$

Differentiation

Definition (Derivative)

Let f be defined on the open set $D \subseteq \mathbb{R}^2$. Then f is *differentiable at* $\vec{x}_0 \in D$ iff there is a vector $\vec{m}$ s.t.

▸ *Picture Time*

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \vec{m} \cdot \vec{h} + \varepsilon \|\vec{h}\|$$

Equivalently: iff there is a vector $\vec{m}$ s.t. for $T(\vec{x}) = f(\vec{x}_0) + \vec{m} \cdot (\vec{x} - \vec{x}_0)$, then

$$\lim_{\vec{x} \to \vec{x}_0} \frac{f(\vec{x}) - T(\vec{x})}{\|\vec{x} - \vec{x}_0\|} = 0$$

Definition (Gradient)

The *gradient* (*vector*) of f, written as ∇f of $\operatorname{grad}(f)$ is

$$\nabla f(\vec{x}_0) = \left\langle \frac{\partial f}{\partial x}\vec{x}_0, \frac{\partial f}{\partial y}\vec{x}_0 \right\rangle$$

Note: ∇ is a vector differential operator (generalizing D_x): $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$.

[3] $T(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Derivative

Nota Bene

f is differentiable[4] at $\vec{a}$ $\Longrightarrow$ $\frac{\partial f}{\partial x}(\vec{a})$ and $\frac{\partial f}{\partial y}(\vec{a})$ both exist

$\frac{\partial f}{\partial x}(\vec{a})$ and $\frac{\partial f}{\partial y}(\vec{a})$ both exist $\not\Longrightarrow$ f is differentiable at $\vec{a}$

Theorem (The "Continuity of Partials Suffices" Thm)

If

1. f_x *and* f_y *exist on* $B(\vec{a};\varepsilon)$ *for some* $\varepsilon > 0$, *and*
2. f_x *and* f_y *are continuous at* $\vec{a}$,

then

1. f *is differentiable at* $\vec{a}$, *and*
2. $f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\vec{x} - \vec{a})$
   where $\varepsilon_1, \varepsilon_2 \to 0$ *as* $x - a_x, y - a_y \to 0$, *resp.*

[4] Careful: Gradient is $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$; Total derivative $f'(\vec{x}_0)$ is $\nabla f(\vec{x}_0)$

Derivative

Proof (The "Continuity of Partials Suffices" Thm).

Let $\vec{a} = \langle x_0, y_0 \rangle$.
NTS: $\Delta f(\vec{a}) = \nabla f(\vec{a}) \cdot \langle \Delta x, \Delta y \rangle + \vec{\varepsilon} \cdot \langle \Delta x, \Delta y \rangle$ with $\vec{\varepsilon} \to \vec{0}$ as $\Delta x, \Delta y \to 0$.

1. Fix y. MVT $\Rightarrow \exists x_1 \in B(x_0; r)$ s.t. $f(x,y) - f(x_0,y) = f_x(x_1,y)(x - x_0)$
2. $f_x \in C(D) \Rightarrow f_x(x_1,y) = f_x(x_0,y_0) + \varepsilon_x$ where $\varepsilon_x \to 0$ as $(x,y) \to (x_0,y_0)$
   So $f(x,y) - f(x_0,y) = [f_x(x_0,y_0) + \varepsilon_x](x - x_0)$ where $\varepsilon_x \underset{x,y \to x_0,y_0}{\longrightarrow} 0$.
3. Fix x. MVT $\Rightarrow \exists y_1 \in B(y_0; r)$ s.t. $f(x,y) - f(x,y_0) = f_y(x,y_1)(y - y_0)$
4. $f_y \in C(D) \Rightarrow f_y(x,y_1) = f_y(x_0,y_0) + \varepsilon_y$ where $\varepsilon_y \to 0$ as $(x,y) \to (x_0,y_0)$
   So $f(x,y) - f(x,y_0) = [f_y(x_0,y_0) + \varepsilon_y](y - y_0)$ where $\varepsilon_y \underset{x,y \to x_0,y_0}{\longrightarrow} 0$.

Whence

$$f(x,y) - f(x_0,y_0) = [f(x,y) - f(x_0,y)] + [f(x_0,y) - f(x_0,y_0)]$$

$$= [f_x(x_0,y_0) + \varepsilon_x](x - x_0) + [f_y(x_0,y_0) + \varepsilon_y](y - y_0)$$

□

Derivatives and Continuity

Theorem ($D \Rightarrow C$ Thm)

If f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.

Proof.

Since f is differentiable at $\vec{a}$,

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{h} + \vec{\varepsilon}\,\|\vec{h}\|$$

where $\vec{\varepsilon} \to 0$ as $\vec{h} \to 0$. Thus

$$\left| f(\vec{a}+\vec{h}) - f(\vec{a}) \right| \le \left| \nabla f(\vec{a}) \cdot \vec{h} \right| + |\vec{\varepsilon}|\,\|\vec{h}\|$$

$$\le \|\nabla f(\vec{a})\|\,\|\vec{h}\| + |\vec{\varepsilon}|\,\|\vec{h}\| = \left(\|\nabla f(\vec{a})\| + |\vec{\varepsilon}| \right) \|\vec{h}\|$$

Whence $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$. □

Algebra of Derivatives

Proposition (Algebra of Derivatives)

Let f and g be differentiable functions at $\vec{a}$. Then

- *$f \pm g$ is differentiable at $\vec{a}$*
- *$f \cdot g$ is differentiable at $\vec{a}$*
- *$f \div g$ is differentiable at $\vec{a}$ as long as $g(\vec{a}) \neq 0$*
- $\nabla(f \pm g) = (\nabla f) \pm (\nabla g)$
- $\nabla(f \cdot g) = (\nabla f)g + f(\nabla g)$
- $\nabla(f \div g) = \dfrac{(\nabla f)g - f(\nabla g)}{g^2}$ *when $g(\vec{a}) \neq 0$*

Proof.

Homework. Pg 462, #14. □

See: §10.2. Problem 4, pg461 (Maple time.)

Directional Derivatives

Thinking Out Loud...

1.
   - f_x is the derivative in the $\langle 1, 0\rangle$ direction
   - f_y is the derivative in the $\langle 0, 1\rangle$ direction
2.
   - $(x_0 + h, y_0) \underset{h\to 0}{\longrightarrow} (x_0, y_0)$ equiv to $\langle x_0, y_0\rangle + h\langle 1, 0\rangle \underset{h\to 0}{\longrightarrow} \langle x_0, y_0\rangle$
   - $(x_0, y_0 + k) \underset{k\to 0}{\longrightarrow} (x_0, y_0)$ equiv to $\langle x_0, y_0\rangle + k\langle 0, 1\rangle \underset{k\to 0}{\longrightarrow} \langle x_0, y_0\rangle$
3. With an arbitrary direction $\vec{u}$ (unit vector): $\vec{x} + h\,\vec{u} \underset{h\to 0}{\longrightarrow} \vec{x}_0$

Definition (Directional Derivative)

Let f be defined on an open set D and $\vec{a} \in D$. Then the *directional derivative of f in the direction of $\vec{u}$, a unit vector*, is given, if the limit is finite, by

$$D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a} + h\,\vec{u}) - f(\vec{a})}{h}$$

or

$$\frac{\partial f}{\partial \vec{u}}(\vec{a}) = \lim_{h\to 0} \frac{f(x + hu_x, y + hu_y) - f(x, y)}{h}$$

Directional Derivative's Properties

Theorem

If f is differentiable at $\vec{a}$, then $D_{\vec{u}}f(\vec{a})$ exists for any direction $\vec{u}$. And

$$D_{\vec{u}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$

Proof.

Simple computation from: $f(\vec{a} + h\vec{u}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (h\vec{u}) + \varepsilon \|h\vec{u}\|$ □

Corollary ("Method of Steepest Ascent/Descent")

Let f be differentiable at $\vec{a}$. Then

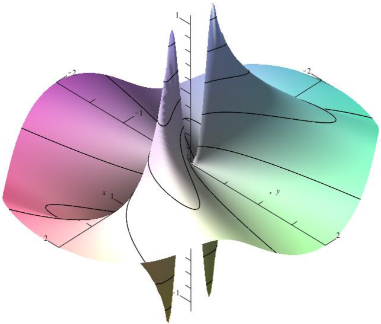
1. *The max rate of change of f at $\vec{a}$ is $\|\nabla f(\vec{a})\|$ in the direction of $\nabla f(\vec{a})$.*
2. *The min rate of change of f at $\vec{a}$ is $-\|\nabla f(\vec{a})\|$ in the direction of $-\nabla f(\vec{a})$.*

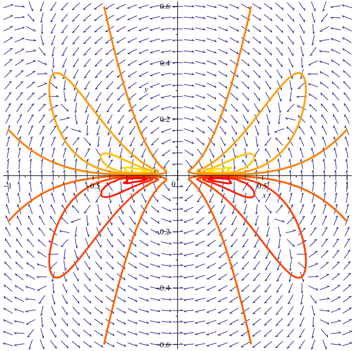
Proof.

Simple computation from: $D_{\vec{u}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u} = \|\nabla f(\vec{a})\| \, \|\vec{u}\| \cos(\theta)$ □

Visit Maple.

Directional Derivative's Weird Properties





Gradient field & contour plot

$$f(x,y) = \frac{x^2 y}{x^6 + y^2}$$

f is not continuous at $\vec{0}$, but has directional derivatives in all directions at $\vec{0}$!

The Chain Rule

Theorem (The Chain Rule)

If $x(t)$ and $y(t)$ are differentiable at t_0, and f is differentiable at $\vec{a} = (x(t_0), y(t_0))$, then f composed with x and y is differentiable at t_0 with

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Proof.

Let $z = f(x, y)$ and $\Delta t = t_1 - t_0$. Then $\Delta x = x(t_1) - x(t_0)$ and $\Delta y = y(t_1) - y(t_0)$. Since f is differentiable, we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

So

$$\frac{\Delta z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Since $\Delta t \to 0 \implies \Delta x, \Delta y \to 0$, then $\varepsilon_1, \varepsilon_2 \to 0$ with Δt. □

The Chain Rule Extended

Corollary (*MCR Corollary)*

If $x(t,s)$ and $y(t,s)$ are differentiable at (t_0,s_0), and $z=f(x,y)$ is differentiable at $\vec{a}=(x(t_0,s_0),y(t_0,s_0))$, then f composed with x and y is differentiable at (t_0,s_0) with

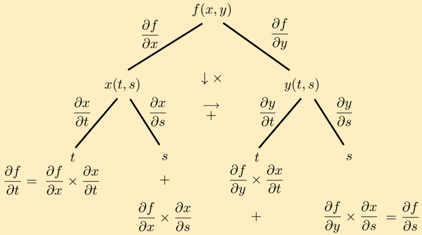
$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad \text{and} \qquad \frac{dz}{ds}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Two Views

$$\begin{bmatrix}\frac{dz}{dt} & \frac{dz}{ds}\end{bmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s}\\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s}\end{bmatrix}$$

$$=\nabla f(x,y)\cdot\frac{\partial(x,y)}{\partial(t,s)}$$

$$=\nabla f(x,y)\cdot J_{(x,y)}(t,s)$$



The Mean Value Theorem

Theorem (*MVT for Two*)

Suppose f is differentiable on the open D containing the segment $L(\vec{p},\vec{q})$. Then there is a $\vec{c}$ on L s.t.

$$f(\vec{p}) - f(\vec{q}) = \nabla f(\vec{c}) \cdot (\vec{p} - \vec{q})$$

Proof.

1. Set $(x_0, y_0) = \vec{q}$ and $(h,k) = \vec{p} - \vec{q}$
2. Set $g(t) = f(x_0 + ht, y_0 + kt)$ for $t \in [0,1]$ (g parametrizes f on L)
3. Then $g(1) - g(0) = g'(\theta)(1-0)$ for some $\theta \in (0,1)$; i.e.
$$f(\vec{p}) - f(\vec{q}) = g'(\theta)$$
4. The MCR implies
$$g'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \langle f_x, f_y \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$$

□

Taylor's Theorem

Theorem (MV Taylor's Theorem)

Suppose f has partial $(n+1)$st derivatives (of all 'mixtures') existing on $B(\vec{a};r)$. Then for $\vec{x} = \vec{a} + (h,k)$ in $B(\vec{a};r)$,

$$\begin{aligned} f(\vec{a}+(h,k)) = & f(\vec{a}) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(\vec{a}) \\ & + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(\vec{a}) + \cdots \\ & + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\vec{a}) + R_n \end{aligned}$$

where

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(\vec{a}+\theta(h,k))$$

for some $\theta \in (0,1)$.

Taylor's Theorem Eg

Example (Second Order, Two Variable)

Find the Taylor polynomial of order 2 at $\vec{a} = \langle 1, 1\rangle$ and remainder for $f(x, y) = x^2 y$ and $\vec{x} = \langle 1, 1\rangle + \langle h, k\rangle$.

1. $$f(\vec{x}) = f(1,1) + \left[f_x(1,1)\cdot h + f_y(1,1)\cdot k\right] + \frac{1}{2}\left[f_{xx}(1,1)\cdot h^2 + 2f_{xy}(1,1)\cdot hk + f_{yy}(1,1)\cdot k^2\right] + \frac{1}{3!}\left[f_{xxx}(1+\theta h, 1+\theta k)\cdot h^3 + 3f_{xxy}(1+\theta h, 1+\theta k)\cdot h^2 k + 3f_{xyy}(1+\theta h, 1+\theta k)\cdot hk^2 + f_{yyy}(1+\theta h, 1+\theta k)\cdot k^3\right]$$
   where $\theta \in (0, 1)$

2. $$f(1+h, 1+k) = 1 + [2h + k] + \frac{1}{2}\left[2h^2 + 4hk + 0k^2\right] + R_2$$
   and $R_2 = \frac{1}{6}\left[0h^3 + 6h^2k + 0hk^2 + 0k^3\right] = h^2 k$ with $\theta \in (0, 1)$

Multiple Integration

Definition (The Double Sums)

Suppose f is bounded on $R = [a, b] \times [c, d]$. Let $P = P_1 \times P_2$ be a partition of R given by $P_1 = \{a = x_0, \ldots, x_n = b\}$ and $P_2 = \{c = y_0, \ldots, y_m = d\}$ with $R_{ij} = [x_{i-1}, y_{j-1}] \times [x_i, y_j]$. Then the area of R_{ij} is $A_{ij} = \Delta x_i \cdot \Delta y_j$

- Set $\|P\| = \max\{\Delta x_i, \Delta y_j\}$.
- Define
$$M_{ij}(f) = \sup_{R_{ij}} f(x, y) \quad \text{and} \quad m_{ij}(f) = \inf_{R_{ij}} f(x, y)$$
- Then define
$$U(P, f) = \sum_i \sum_j M_{ij}\, \Delta x_i \Delta y_j = \sum_{i,j} M_{ij} A_{ij}$$
$$L(P, f) = \sum_i \sum_j m_{ij}\, \Delta x_i \Delta y_j = \sum_{i,j} m_{ij} A_{ij}$$
$$S(P, f) = \sum_i \sum_j f(c_i, d_j)\, \Delta x_i \Delta y_j = \sum_{i,j} f(c_i, d_j) A_{ij}$$
where $(c_i, d_j) \in R_{ij}$ is arbitrary.

A Useful Lemma

Lemma

Let f be bounded on the rectangle R with partition P. Set

$$m = \inf_R f(x,y) \quad \text{and} \quad M = \sup_R f(x,y).$$

1. *Then*

   $$m(b-a)(d-c) \leq L(P,f) \leq S(P,f) \leq U(P,f) \leq M(b-a)(d-c)$$

2. *If Q partitions R and $P \subseteq Q$, then*

   $$L(P,f) \leq L(Q,f) \quad \text{and} \quad U(Q,f) \leq U(P,f)$$

3. *For any partitions P and Q of R,* $L(P,f) \leq U(Q,f)$.

4. $\sup_P L(P,f) \leq \inf_P U(P,f)$

5. *The area of R is* $A = \sum_{ij} A_{ij} = (b-a)(d-c)$

The Integral

Definition (Double Integral)

Let f be bounded on the rectangle R. Then f is *Riemann integrable on* R iff the *upper double integral* and the *lower double integral*, resp.,

$$\overline{\iint_R} f\,dA = \inf_P U(P,f) \quad \text{and} \quad \underline{\iint_R} f\,dA = \sup_P L(P,f)$$

both exist and are equal. We write $\iint_R f\,dA$ for the common value.

Theorem

A bounded function f on the rectangle R is Riemann integrable iff

1. *for any $\varepsilon > 0$ there is a partition P of R s.t.*

$$U(P,f) - L(P,f) < \varepsilon.$$

2. *there is a seq of partitions $\{P_n\}$ s.t.*

$$\lim_{n\to\infty} U(P_n,f) = I = \lim_{n\to\infty} L(P_n,f).$$

A Sample

Example

Find $\displaystyle\iint_R f\,dA$ when $f(x,y) = \frac{1}{2}\sin(x+y)$ and $R = [0, \frac{\pi}{2}]^2$.

1. Use a uniform grid: $x_i = \frac{i}{n}\frac{\pi}{2}$, $y_j = \frac{j}{n}\frac{\pi}{2}$, & $(c_i, d_j) = (x_i, y_j)$ for $i, j = 0..n$
2. A generic Riemann sum becomes

$$S(P_n, f) = \sum_{i,j\in[1,n]} f\left(\frac{i}{n}\frac{\pi}{2}, \frac{j}{n}\frac{\pi}{2}\right)\left(\frac{i}{n}\frac{\pi}{2} - \frac{i-1}{n}\frac{\pi}{2}\right)\left(\frac{j}{n}\frac{\pi}{2} - \frac{j-1}{n}\frac{\pi}{2}\right)$$

$$= \frac{\pi^2}{4n^2}\sum_{i,j\in[1,n]} \frac{1}{2}\sin\left(\frac{i}{n}\frac{\pi}{2} + \frac{j}{n}\frac{\pi}{2}\right)$$

3. Since $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, we have

$$S(P_n, f) = \frac{\pi^2}{8n^2}\sum_{i,j\in[1,n]}\left[\sin\left(\frac{i}{n}\frac{\pi}{2}\right)\cos\left(\frac{j}{n}\frac{\pi}{2}\right) + \cos\left(\frac{i}{n}\frac{\pi}{2}\right)\sin\left(\frac{j}{n}\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi^2}{8n^2}\sum_{i,j\in[1,n]}\left[\sin\left(\frac{i}{n}\frac{\pi}{2}\right)\cos\left(\frac{j}{n}\frac{\pi}{2}\right)\right] + \sum_{i,j\in[1,n]}\left[\cos\left(\frac{i}{n}\frac{\pi}{2}\right)\sin\left(\frac{j}{n}\frac{\pi}{2}\right)\right]$$

A Sample (cont)

Example (cont)

4. Distribute the sums

$$S(P_n, f) = \frac{\pi^2}{8n^2}\left[\sum_{i=1}^{n} \sin\left(\frac{i}{n}\frac{\pi}{2}\right)\sum_{j=1}^{n}\cos\left(\frac{j}{n}\frac{\pi}{2}\right) + \sum_{i=1}^{n}\cos\left(\frac{i}{n}\frac{\pi}{2}\right)\sum_{j=1}^{n}\sin\left(\frac{j}{n}\frac{\pi}{2}\right)\right]$$

$$= 2\frac{\pi^2}{8n^2}\sum_{i=1}^{n}\cos\left(\frac{i}{n}\frac{\pi}{2}\right)\sum_{j=1}^{n}\sin\left(\frac{j}{n}\frac{\pi}{2}\right)$$

$$= \left[\frac{\pi}{2n}\sum_{i=1}^{n}\cos\left(\frac{i}{n}\frac{\pi}{2}\right)\right]\cdot\left[\frac{\pi}{2n}\sum_{j=1}^{n}\sin\left(\frac{j}{n}\frac{\pi}{2}\right)\right]$$

5. $\lim_{n\to\infty} \frac{\pi}{2n}\sum_{j=1}^{n} T\left(\frac{j}{n}\frac{\pi}{2}\right) = \int_0^{\pi/2} T(x)\,dx$, so

$$\lim_{n\to\infty} S(P_n, f) = \int_0^{\pi/2}\cos(x)\,dx \cdot \int_0^{\pi/2}\sin(x)\,dx = 1$$

6. Whence $\iint_{[0,\pi/2]\times[0,\pi/2]} \frac{1}{2}\sin(x+y)\,dA = 1$

Continuous Functions

Theorem (Continuous Functions Are Integrable)

If f is continuous on $R = [a, b] \times [c, d]$, then f is integrable on R.

Proof.

Let $\varepsilon > 0$. Set $A = \text{area}(R)$.

1. Since f is cont on R, then f is unif cont on R. Hence there is a $\delta > 0$ s.t. whenever $\vec{x_1}, \vec{x_2} \in R$ with $\|\vec{x_1} - \vec{x_2}\| < \delta$, then $|f(\vec{x_1}) - f(\vec{x_2})| < \varepsilon$.
2. Choose a partition P s.t. $\|P\| < \delta$.
3. Then $U(P, f) - L(P, f) = \sum_{i,j} M_{ij} \Delta x_i \Delta y_j - \sum_{i,j} m_{ij} \Delta x_i \Delta y_j$. I.e.,

$$U(P, f) - L(P, f) = \sum_{i,j} (M_{ij} - m_{ij}) \Delta A_{ij} < \sum_{i,j} \varepsilon \Delta A_{ij} = A\,\varepsilon$$

□

Bilinearity

Theorem (Bilinearity of Integration)

1. *Let f_1 and f_2 be integrable on R, and c_1 and c_2 be constants. Then*
$$\iint_R c_1 f_1 \pm c_2 f_2 \, dA = c_1 \iint_R f_1 \, dA \pm c_2 \iint_R f_2 \, dA$$
2. *Let f be bounded on $R = R_1 + R_2$.*
   2.1 *Then f is integrable on R iff f is integrable on R_1 and R_2.*
   2.2 *If f is integrable on R, then*
$$\iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA$$

Proposition

Let f be integrable on R with $m = \min_R f$ and $M = \max_R f$. Then
$$m \cdot \mathrm{area}(R) \leq \iint_R f \, dA \leq M \cdot \mathrm{area}(R)$$

Iteration

Thinking Out Loud...

1. Fix x^*. Suppose $f(x^*, y)$ is an integrable function of y. Define

$$g(x) = \int_{[c,d]} f(x, y)\, dy$$

   Then integrate g to get

$$\int_{[a,b]} \left[\int_{[c,d]} f(x, y)\, dy \right] dx$$

2. Fix y^*. Suppose $f(x, y^*)$ is an integrable function of x. Define

$$h(y) = \int_{[a,b]} f(x, y)\, dx$$

   Then integrate h to get

$$\int_{[c,d]} \left[\int_{[a,b]} f(x, y)\, dx \right] dy$$

How do these integrals relate to $\iint_R f\, dA$?

Iteration and Guido Fubini

Theorem (Fubini (1910))

Let f be integrable on a rectangle R. If for each x, the function $h(y) = f(x, y)$ is integrable over $y \in [c, d]$, then $g(x) = \int_c^d f(x, y)\, dy$ is integrable for $x \in [a, b]$, and

$$\iint_R f\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

Corollary

Let f be integrable on a rectangle R. If

1. $h(y) = f(x, y)$ *is integrable over* $y \in [c, d]$, *and*
2. $k(x) = f(x, y)$ *is integrable over* $x \in [a, b]$,

then

$$\iint_R f\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Proving Fubini's Theorem

Proof (sketch).

Let $\varepsilon > 0$.

1. Find a partition P of $[a,b] \times [c,d]$ where $U(P,f) - L(P,f) < \varepsilon$
2. 'Slice' this partition into $P_1(x) \times P_2(y)$.
3. Use $U(P_1,g) - L(P_1,g) < U(P,f) - L(P,f)$ to show

   $g(x) = \displaystyle\int_{[c,d]} f(x,y)dy$ is integrable over $[a,b]$.
4. Show $L(P,f) \leq \displaystyle\int_{[a,b]} g\,dx \leq U(P,f)$
5. Conclude $\displaystyle\int_{[a,b]} g(x)\,dx = \iint_R f(x,y)dA$
6. Use symmetry to have $\displaystyle\int_{[c,d]} h(y)\,dy = \iint_R f(x,y)dA$

Observe the doneness of the proof. $\square$

Fubini Examples

Example (*Good Function! Biscuit!*)

Let $N(x, y) = e^{-(x^2+y^2)}$ and $R = \mathbb{R}^2$.

1. Change to polar coordinates.

$$\iint_R N(x, y)\, dA = \iint_{[0,\infty]\times[0,2\pi]} N(r, \theta)\, dA$$

2. Apply Fubini's thm two ways:

2.1 $\displaystyle\iint_R N(r, \theta)\, dA = \int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} r\, dr\right] d\theta = \int_0^{2\pi} \tfrac{1}{2}\, d\theta = \pi$

2.2 $\displaystyle\iint_R e^{-x^2} e^{-y^2} dA = \int_{-\infty}^{\infty} e^{-y^2}\left[\int_{-\infty}^{\infty} e^{-x^2} dx\right] dy = \int_{-\infty}^{\infty} e^{-y^2} dy \cdot \int_{-\infty}^{\infty} e^{-x^2} dx$

3. Whence $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Whereupon $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} dx = 1$.

Fubini Examples II

Example (*Bad Function! No Biscuit!*)

Let $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on $R = [0,1] \times [0,1]$.

1. $\displaystyle\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy = -\frac{\pi}{4}$
2. $\displaystyle\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx = +\frac{\pi}{4}$
3. $\displaystyle\int_0^1 \left[\int_0^1 |f(x,y)|\,dy\right] dx = \infty$

So $\displaystyle\iint_R f(x,y)\,dA$ does not exist

The Leibniz Rule

Theorem (Leibniz Rule)

Suppose f *has continuous partials on* $R = [a,b] \times [c,d]$*. Set* $g(x) = \int_c^d f(x,y)\,dy$*. Then* g *is differentiable on* (a,b) *and*

$$\frac{d}{dx}g(x) = \int_c^d \frac{\partial}{\partial x} f(x,y)\,dx$$

Proof.

1. f has cont partials $\implies$ f is cont and differentiable on $\operatorname{int}(R)$
2. Then f is integ., so for every fixed x^*, $f(x^*,y)$ is integ. on $[c,d]$
3. Choose $x \neq x^*$, then $\exists x_0$ between x and x^* s.t.

$$\frac{g(x)-g(x^*)}{x-x^*} = \int_c^d \frac{f(x,y)-f(x^*,y)}{x-x^*}\,dy = \int f_x(x_0,y)\,dy$$

4. Take limits as $x \to x^*$ to finish

□

Camille Jordan's Content

Definition (Jordan Content Zero)

A set S has *Jordan content zero* iff for each $\varepsilon > 0$ there is a finite collection $\mathcal{R}$ of rectangles R_{ij} s.t.

- $S \subseteq \bigcup_{ij} R_{ij}$
- $\text{area}(\mathcal{R}) = \sum_{ij} \text{area}(R_{ij}) < \varepsilon$

A bounded set D is *Jordan measurable* iff ∂D has Jordan content zero.

Examples

- log spiral on $[9.5297^{-1}, 9.5297]$
- unit disk
- Hilbert's plane filling curve, space filling curve

Proposition

- *Rectifiable curves have Jordan content zero.*
- *The union of sets of content zero has content zero.*

Jordan's Extension

Theorem

If f is continuous on $R = [a, b] \times [c, d]$ except on a set of Jordan content zero, then f is integrable on R.

Proof.

1. Since R is compact and f is cont, $\exists M > 0$ s.t. $|f(x, y)| < M$ on R.
2. For each R_{ij} we see $M_{ij} - m_{ij} < 2M$.
3. Let S be the set of discontinuities of f. So S has content zero.
4. Let $\varepsilon > 0$. Find P s.t. for the rect's covering S, the $\sum \text{area}(R_{ij}) < \varepsilon$
5. Divide the P into P_S and $P_{\bar{S}}$ where P_S contains the rectangles covering S. Then $U(P) - L(P) = [U(P_S) + U(P_{\bar{S}})] - [L(P_S) + L(P_{\bar{S}})]$.
6. Combine with 4: $U(P_S) - L(P_S) \leq \sum (M_{ij} - m_{ij})\Delta A_{ij} < 2M\varepsilon$
7. f is unif cont on $P_{\bar{S}}$ so refine P to obtain $M_{ij} - m_{ij} < \varepsilon$ on P'
8. Then $\displaystyle\sum_{R_{ij} \in P'} (M_{ij} - m_{ij})\Delta A_{ij} < \varepsilon \sum \Delta A_{ij} < \varepsilon A$

□

Bounded, Jordan-Measurable Regions

Proposition (Integral on a B'nded, Jordan-Mble Set)

Let D be a bounded, Jordan-measurable region in $\mathbb{R}^2$ and let f be continuous on D. Define $\chi_D(x) = 1$ for $x \in D$ and 0 for $x \notin D$. Suppose the rectangle $R \supset D$.

- $\iint_D f\,dA \triangleq \iint_R f\,\chi_D\,dA$
- *If D is the region $[a,b] \times [\alpha(x), \beta(x)]$ where $\alpha \leq \beta$, then*

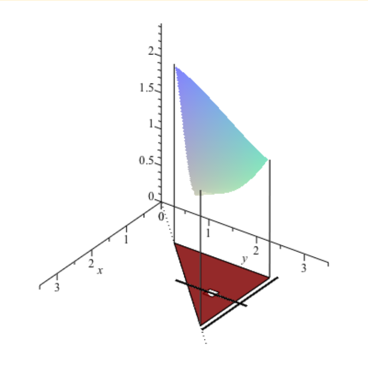
$$\iint_D f\,dA \triangleq \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx$$

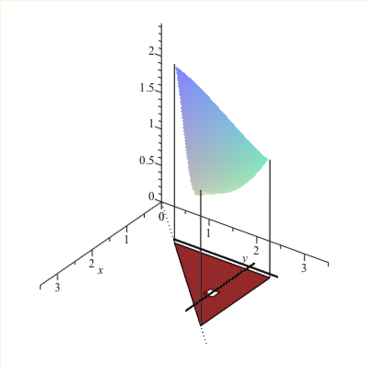
- *If D is the region $[\alpha(y), \beta(y)] \times [c,d]$ where $\alpha \leq \beta$, then*

$$\iint_D f\,dA \triangleq \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx\,dy$$

Dirichlet's Formula

Dirichlet ⊂ Fubini





$$\int_a^b \int_x^b f(x,y)\,dy\,dx \qquad\qquad \int_a^b \int_a^y f(x,y)\,dx\,dy$$

Line Integrals

Definition (Line Integral)

If f is continuous on a region D containing a smooth curve C, then the *line integral of* f *along* C is

$$\int_C f\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_i, d_i)\,\Delta s_i$$

Proposition

If C *has a smooth parametrization* $(x(t), y(t))$ *for* $t \in [a, b]$*, then*

$$\begin{aligned}\int_C f\,ds &= \int_a^b f(x(t), y(t))\,s'(t)\,dt \\ &= \int_a^b f(x(t), y(t))\,\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt\end{aligned}$$

Line Integrals Are Linear

Proposition (Algebraic Properties)

1. $\displaystyle\int_{-C} f\,ds = -\int_C f\,ds$
2. $\displaystyle\int_C f\,ds = \sum_{i=1}^{n}\int_{C_i} f\,ds$ *where* $C = \bigcup_i C_i$
3. $\displaystyle\left|\int_C f\,ds\right| \leq ML$ *where* $L = \mathrm{length}(C)$ *&* $M \geq \max_C |f(x,y)|$.

Examples

1. $\int_C xy\,dx + (x^2+y^2)dy$ with C the unit circle in the 1st quadrant
2. $\int_C x\,ds$ with C the unit circle in the 1st quadrant
3. $\int_S xy\,dx + (x^2+y^2)dy$ with S being the unit square having the vertex set $[(1,0),(1,1),(0,1),(0,0)]$

Green's Theorem

Theorem (Green's Theorem[5])

Let D be a simple region in $\mathbb{R}^2$ with a positively-oriented, closed boundary ∂D. If $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ is a continuously differentiable vector field on an open region containing D, then

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D (N_x - M_y)dx\,dy$$

Theorem (Differential Forms Version)

For D as above and a differentiable $(n-1)$-form ω, $\displaystyle\int_{\partial D} \omega = \int_D d\omega$

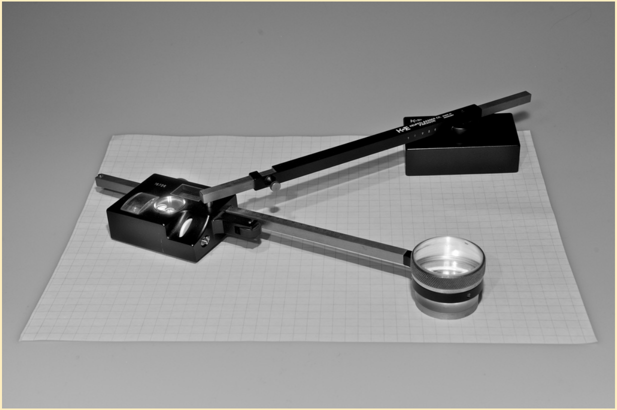
Corollary (Area of a Region)

For f and D as above, $\displaystyle\mathrm{Area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx.$

[5] There are a number of equivalent forms of Green's Theorem.

Interlude

Green's Theorem Applied[6]



A Planimeter

[6] Build your own planimeter.

Proving Green's Theorem

Proof.

I. $D = \{(x, y) : a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)\}$. By linearity, NTS:

$$\oint_{\partial D} M\,dx = -\iint_D M_y \quad \text{and} \quad \oint_{\partial D} N\,dy = \iint_D N_x$$

1. Now $\displaystyle\iint_D M_y = \int_a^b \int_{g_1}^{g_2} M_y\,dy\,dx.$
2. The FToC gives $\displaystyle\iint_D M_y = \int_a^b [M(x, g_2) - M(x, g_1)]dx$
3. Decompose ∂D into $D_1 = \{x, g_1(x)\}$, $D_2 = \{x = b, g_1(b) \leq y \leq g_2(b)\}$, $D_3 = \{x, g_2(x)\}$, and $D_4 = \{x = a, g_2(a) \geq y \geq g_1(a)\}$
4. On D_2 and D_4, $dx = 0$, so $\oint_{\partial D} = \oint_{D_1} + \oint_{D_3}$
5. Then $\displaystyle\oint_{\partial D} M\,dx = \int_a^b M(t, g_1(t))\,dt + \int_b^a M(t, g_2(t))\,dt$
   $\displaystyle= \int_a^b M(t, g_1(t)) - M(t, g_2(t))\,dt = -\iint_D M_y.$ Aha! $\displaystyle\oint_{\partial D} M\,dx = -\iint_D M_y.$

II. Analogously, $\displaystyle\oint_{\partial D} N\,dy = \iint_D N_x.$ □

Forms of Green's Theorem

Theorem

"Under suitable conditions,"

1. $\displaystyle\oint_{\partial D} M\,dx + N\,dy = \oint_{\partial D} \vec{F}\cdot\vec{T}\,ds$ *Circulation Thm*

2. $\displaystyle\oint_{\partial D} M\,dx - N\,dy = \oint_{\partial D} \vec{F}\cdot\vec{N}\,ds$ *Flux Thm*

3. $\displaystyle\iint_D (M_x + N_y)\,dA = \iint_D \operatorname{div}(\vec{F})\,dA$ *Divergence Thm*

4. $\displaystyle\iint_D (N_x - M_y)\,dA = \iint_D \operatorname{curl}(\vec{F})\,dA$ *Curl Thm*

$\operatorname{div}(\vec{v}) = \nabla\cdot\vec{v}$ *and* $\operatorname{curl}(\vec{v}) = \nabla\times\vec{v}$

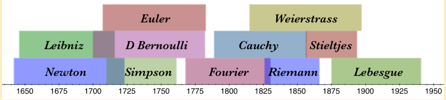
Introduction to Lebesgue Measure

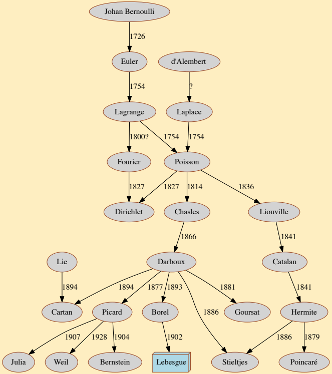
Prelude

There were two problems with calculus: there are functions where

- $f(x) \neq \int f'(x)\,dx$
- $f(x) \neq \dfrac{d}{dx}\left[\int f(x)\,dx\right]$

In his 1902 dissertation, "Intégrale, longueur, aire," Lebesgue wrote, "It thus seems to be natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible."





Henri Lebesgue's Mathematical Genealogy (partial)

What's in a Measure

Goals

THE BEST *measure* would be a real-valued set function μ that satisfies

1. $\mu(I) = \mathrm{length}(I)$ where I is an interval
2. μ is *translation invariant*: $\mu(x+E) = \mu(E)$ for any $x \in \mathbb{R}$
3. if $\{E_n\}$ is pairwise disjoint, then $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$
4. $\mathrm{dom}(\mu) = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$)

THE BAD NEWS:

$$\left\{ \begin{array}{c} \textit{continuum hypothesis} \\ + \quad \textit{axiom choice} \end{array} \right\} \Longrightarrow \text{1, 3, and 4 are incompatible}$$

THE PLAN:

- Give up on 4. (cf. *Vitali*)
- 1. and 2. are nonnegotiable
- Weaken 3., then reclaim it

Sigma Algebras

Definition

Sigma Algebra of Sets

Algebra: A collection of sets $\mathcal{A}$ is an *algebra* iff $\mathcal{A}$ is closed under unions and complements.

σ-Algebra: An algebra of sets $\mathcal{A}$ is a *σ-algebra* iff $\mathcal{A}$ is closed under countable unions.

Proposition

Let $\mathcal{A}$ be a nonempty algebra of sets of reals. Then

- $\emptyset$ *and* $\mathbb{R} \in \mathcal{A}$.
- $\mathcal{A}$ *is closed under intersection.*

Let $\mathcal{A}$ be a nonempty σ-algebra of sets of reals. Then

- $\mathcal{A}$ *is closed under countable intersections.*

Sigma Samples

Examples

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$

2. $\mathcal{F} = \{F \subset \mathbb{R} : F \text{ is finite or } F^c \text{ is finite}\}$
   2.1 $\mathcal{F}$ is an algebra, the *co-finite algebra*
   2.2 $\mathcal{F}$ is not a σ-algebra
   For each $r \in \mathbb{Q}$, the set $\{r\} \in \mathcal{F}$. But $\bigcup_{r\in\mathbb{Q}}\{r\} = \mathbb{Q} \notin \mathcal{F}$

3. Let $\mathcal{A} = \{\emptyset, [-1, 1], (-\infty, -1) \cup (1, \infty), \mathbb{R}\}$. Is $\mathcal{A}$ an algebra?

4. Any intersection of σ-algebras is a σ-algebra

5. Let $\mathcal{B}(\mathbb{R})$ be the smallest σ-algebra containing all the open sets, the *Borel* σ*-algebra*.

Outer Measure

Definition (Lebesgue Outer Measure)

Let $E \subset \mathbb{R}$. Define the *Lebesgue Outer Measure* μ^* of E to be

$$\mu^*(E) = \inf_{E \subset \bigcup I_n} \sum_n \ell(I_n),$$

the infimum of the sums of the lengths of open interval covers of E.

Proposition (Monotonicity)

If $A \subseteq B$*, then* $\mu^*(A) \leq \mu^*(B)$*.*

Proposition

If I *is an interval, then* $\mu^*(I) = \ell(I)$*.*

Outer Measure of an Interval

Proof.

I. I is closed and bounded (compact). Then $I = [a, b]$.

1. For any $\varepsilon > 0$, $[a, b] \subset (a - \varepsilon, b + \varepsilon)$. So $\mu^*(I) \leq b - a + 2\varepsilon$. Since ε is arbitrary, $\mu^*(I) \leq b - a$.
2. Let $\{I_n\}$ cover $[a, b]$ with open intervals. There is a finite subcover for $[a, b]$. Order the subcover so that consecutive intervals overlap. Then

$$\sum_N \ell(I_k) = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_N - a_N)$$

Rearrange

$$\begin{aligned}\sum_N \ell(I_k) &= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \cdots - (a_2 - b_1) - a_1 \\ &\geq b_N - a_1 > b - a\end{aligned}$$

Whence $\mu^*(I) = b - a$.

Outer Measure of an Interval, II

Proof (cont).

II. Let I be any bounded interval and $\varepsilon > 0$.

1. There is a closed interval $J \subset I$ so that $\ell(I) - \varepsilon < \ell(J)$. Then

$$\ell(I) - \varepsilon < \ell(J) = \mu^*(J) \leq \mu^*(I) \leq \mu^*(\bar{I}) = \ell(\bar{I}) = \ell(I)$$

III. Suppose I is infinite.

1. Then for each n, there is a closed interval $J \subset I$ s.t. $\ell(J) = n$
2. Thence $\mu^*(I) \geq n$ for all n.

Aha! $\mu^*(I) = \infty$

Proposition

$$\mu^*(\mathbb{Q}) = 0$$

Proof.

Order $\mathbb{Q}$ as $\{r_1, r_2, \dots\}$. $\{I_n = (r_n - \varepsilon/2^n, r_n + \varepsilon/2^n)\}$ covers $\mathbb{Q}$ □

Countable Subadditivity

Theorem (μ^* is Countably Subadditive)

Let $\{E_n\}$ *be a countable set sequence in* $\mathbb{R}$. *Then* $\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$

Proof.

I. If $\mu^*(E_n) = \infty$ for any n, then done.

II. Let $\varepsilon > 0$

1. For each n find a cover $\{I_{n,j}\}_{n\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} \ell(I_{n,j}) < \mu^*(E_n) + \frac{\varepsilon}{2^n}$
2. Then $\{I_{n,j}\}_{n,j\in\mathbb{N}}$ covers $E = \bigcup_n E_n$.
3. Whereupon

$$\mu^*(E) \leq \sum_{n,j\in\mathbb{N}} \ell(I_{n,j}) = \sum_{n\in\mathbb{N}} \left[\sum_{j\in\mathbb{N}} \ell(I_{n,j})\right]$$

$$< \sum_{n\in\mathbb{N}} \left[\mu^*(E_n) + \frac{\varepsilon}{2^n}\right] = \sum_{n\in\mathbb{N}} [\mu^*(E_n)] + \varepsilon$$

□

Open Holding & Lebesgue's Measure

Corollary

Given $E \subseteq \mathbb{R}$ *and* $\varepsilon > 0$, *there is an open set* $O \supseteq E$ *s.t.*

$$\mu^*(E) \le \mu^*(O) \le \mu^*(E) + \varepsilon$$

Definition (Carathéodory's Condition)

A set E is *Lebesgue measurable* iff for every (test) set A,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Let $\mathfrak{M}$ be the collection of all Lebesgue measurable sets.

Corollary

For any A *and* E,

$$\mu^*(A) = \mu^*\big((A \cap E) \cup (A \cap E^c)\big) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Much Ado About Nothing

Theorem

If $\mu^*(E) = 0$, then $E \in \mathfrak{M}$; i.e., E is measurable.

Proof.

Given the previous corollary, we need only show that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

1. Since $A \cap E \subset E$, then $\mu^*(A \cap E) \leq \mu^*(E) = 0$.
2. Since $A \cap E^c \subset A$, then $\mu^*(A \cap E^c) \leq \mu^*(A)$.

Whence $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq 0 + \mu^*(A) = \mu^*(A)$. □

Corollary

$$\mu^*(\mathbb{Q}) = 0 \implies \mathbb{Q} \in \mathfrak{M}$$

Unions Work

Theorem

A finite union of measurable sets is measurable.

Proof.

Let E_1 and $E_2 \in \mathfrak{M}$. Let A be a test set.

1. Use $A \cap E_1^c$ as a test set for E_2 which is measurable. Thence
$$\mu^*(A \cap E_1^c) = \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c)$$
2. Note $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$. Whereupon
$$\begin{aligned}
&\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c) \\
&\qquad = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_2^c)) \\
&\qquad \leq \left[\mu^*(A \cap E_1) + \mu^*(A \cap E_2 \cap E_1^c)\right] + \mu^*(A \cap E_1^c \cap E_2^c) \\
&\qquad \leq \mu^*(A \cap E_1) + \left[\mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)\right] \\
&\qquad = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \\
&\qquad = \mu^*(A)
\end{aligned}$$

□

Countable Unions Work

Theorem

The countable union of measurable sets is measurable.

Proof.

Let $E_k \in \mathfrak{M}$ and $E = \bigcup_n E_n$. Choose a test set A.
We need to show $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$.

1. Set $F_n = \bigcup^n E_k$ and $F = \bigcup^\infty E_k = E$. Define $G_1 = E_1$, $G_2 = E_2 - E_1, \ldots, G_k = E_k - \bigcup^{k-1} E_j$, and $G = \bigcup G_k$. Then

   (i) $G_i \cap G_j = \emptyset, \ (i \neq j)$ (ii) $F_n = \bigcup^{n} G_k$ (iii) $F = G = E$

2. Test F_n with A to obtain $\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$
3. Test G_n with $A \cap F_n$ to obtain

$$\begin{aligned}\mu^*(A \cap F_n) &= \mu^*((A \cap F_n) \cap G_n) + \mu^*((A \cap F_n) \cap G_n^c) \\ &= \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})\end{aligned}$$

Countable Unions Work, II

Proof.

4. Iterate $\mu^*(A \cap F_n) = \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})$ from 3 to have
$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap G_k)$$
5. Since $F_n \subseteq F$, then $F^c \subseteq F_n^c$ for all n, then
$$\mu^*(A \cap F_n^c) \geq \mu^*(A \cap F^c)$$
6. Whence
$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap G_k) + \mu^*(A \cap F^c)$$
The summation is increasing & bounded, so convergent.
7. However
$$\sum_{k=1}^{\infty} \mu^*(A \cap G_k) \geq \mu^*\left(\bigcup_{k=1}^{\infty}(A \cap G_k)\right) = \mu^*(A \cap F)$$

Aha! $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c)$ □

Everything Works

Corollary

The collection of Lebesgue measurable sets $\mathfrak{M}$ is a σ-algebra.

Corollary

The Borel sets are measurable. (There are measurable, non-Borel sets.)

$$\mathcal{B}(\mathbb{R}) \subsetneqq \mathfrak{M} \subsetneqq \mathcal{P}(\mathbb{R})$$

Definition (Lebesgue Measure)

Lebesgue measure μ is μ^* restricted to $\mathfrak{M}$. So $\mu : \mathfrak{M} \to [0, \infty]$.

Definition (Almost Everywhere)

A property P holds *almost everywhere* (a.e.) iff $\mu\big(\{x : \neg P(x)\}\big) = 0$.

The Return of Additivity

Theorem

Let $\{E_n\}$ be a countable (finite or infinite) sequence of pairwise disjoint sets in $\mathfrak{M}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Proof.

I. n is finite.

1. For $n = 1$, ✓
2. $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = E_n$ and $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c = \bigcup_{k=1}^{n-1} E_k$
3. $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n\right) + \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^c\right)$
   $= \mu(E_n) + \mu\left(\bigcup_{k=1}^{n-1} E_k\right) = \mu(E_n) + \sum_{k=1}^{n-1} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k)$

II. n is infinite.

1. $\bigcup_{k=1}^{n} E_k \subset \bigcup_{k=1}^{\infty} E_k \implies \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
2. A bnded & incr sum converges. Thus $\sum_{k=1}^{\infty} \mu(E_k) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
3. Subadditivity finishes the proof.

□

Adding an Example

Example

Set $E_n = \left(\frac{n-1}{n}, \frac{n}{n+1} \right)$ for $n = 1..\infty$.

1. The E_n are pairwise disjoint.
2. $\mu(E_n) = \ell(E_n) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$
3. $\mu\left(\bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right]$

Whence $\mu\left(\bigcup_{n=1}^{\infty} E_n \right) = 1$.

NOTA BENE: $\bigcup_{n=1}^{\infty} E_n = (0,1) - \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$. Hence $\bigcup_{n=1}^{\infty} E_n = (0,1)$ *a.e.*

Matryoshka

Theorem

If $\{E_n\}$ is a seq of nested, measurable sets with $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof.

1. Set $E = \bigcap_{k=1}^{\infty} E_k$. Set $F_k = E_k - E_{k+1}$. The F_k are pairwise disjoint.
2. Since $\bigcup_{k=1}^{\infty} F_k = E_1 - E$, then $\mu(E_1 - E) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \mu(E_k - E_{k+1})$.
3. If $A \subset B$, then $\mu(A - B) = \mu(A) - \mu(B)$. Apply to the formula above.
4. $\mu(E_1) - \mu(E) = \sum_{k=1}^{\infty} \mu(E_k) - \mu(E_{k+1}) = \mu(E_1) - \lim_{k\to\infty} \mu(E_k)$

Since $\mu(E_1)$ is finite, we're done. □

The Cantor Set

Cantor Sets[7]

I. Constructing C

1. Set $C_0 = [0,1]$
2. Set $C_1 = C_0 - (\frac{1}{3}, \frac{2}{3})$
3. Set $C_2 = C_1 - (\frac{1}{3^2}, \frac{2}{3^2}) - (\frac{7}{3^2}, \frac{8}{3^2})$
4. Set $C_3 = C_2 - (\frac{1}{3^3}, \frac{2}{3^3}) - (\frac{7}{3^3}, \frac{8}{3^3}) - (\frac{19}{3^3}, \frac{20}{3^3}) - (\frac{25}{3^3}, \frac{26}{3^3})$
5. Let $C = \bigcap C_i$

II. Properties of C

1. $\mu(C_0) = 1$, $\mu(C_1) = 2/3$, $\mu(C_2) = 4/9$, $\mu(C_3) = 8/27$, …So $\mu(C_n) = \frac{2}{3}\mu(C_{n-1}) = \frac{2^n}{3^n}$ Whence $\mu(C) = 0$.
2. C is uncountable
3. C is perfect
4. C is nowhere dense
5. C is compact
6. C is totally disconnected
7. $(\forall i)\ \partial C_i \subset C$
8. $(\forall i)\ \frac{1}{4} \notin \partial C_i$, but $\frac{1}{4} \in C$

[7] Cantor gave the set in a footnote to show "perfect" $\not\subset$ "everywhere dense".

Not So Strange After All

Theorem

Let $E \subseteq \mathbb{R}$ and let $\varepsilon > 0$. TFAE:

1. *E is measurable*
2. *There is an open set $O \supset E$ s.t. $\mu^*(O - E) < \varepsilon$*
3. *There is a closed set $F \subset E$ s.t. $\mu^*(E - F) < \varepsilon$*

Proposition

Let S and T be measurable subsets of $\mathbb{R}$. Then

$$\mu(S \cup T) + \mu(S \cap T) = \mu(S) + \mu(T)$$

Functionally Measurable

Theorem (Measurability Conditions for Functions)

Let $f : D \to \mathbb{R}_\infty$ for some $D \in \mathfrak{M}$. TFAE

1. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((r,\infty)\big)$ is measurable.*
2. *For each $r \in \mathbb{R}$, the set $f^{-1}\big([r,\infty)\big)$ is measurable.*
3. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty,r)\big)$ is measurable.*
4. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty,r]\big)$ is measurable.*

Proof.

$1 \Rightarrow 2$: $\{x \mid f(x) \geq r\} = \bigcap_n \{x \mid f(x) > r - 1/n\}$

$2 \Rightarrow 3$: $\{x \mid f(x) < r\} = D - \{x \mid f(x) \geq r\}$

$3 \Rightarrow 4$: $\{x \mid f(x) \leq r\} = \bigcap_n \{x \mid f(x) < r + 1/n\}$

$4 \Rightarrow 1$: $\{x \mid f(x) > r\} = D - \{x \mid f(x) \leq r\}$ $\square$

The Measurably Functional

Corollary

If f satisfies any measurability condition, then $\{x \mid f(x) = r\}$ is measurable for each r.

Definition (Measurable Function)

If a function $f : D \to \mathbb{R}_\infty$ has measurable domain D and satisfies any of the measurability conditions, then f is *measurable*.

Definition

Step function: $\phi : [a, b] \to \mathbb{R}_\infty$ is a *step function* if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ s.t. ϕ is constant on each interval $I_k = (x_{k-1}, x_k)$, then

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{I_k}(x)$$

Simple function: A function ψ with range $\{a_1, a_2, \ldots, a_n\}$ where each set $\psi^{-1}(a_k)$ is measurable is a *simple function*.

Simply Stepping

Proposition

Step functions and simple functions are measurable

Theorem (Algebra of Measurable Functions)

Let f and g be measurable on a common domain D, and let $c \in \mathbb{R}$. Then

1. $f + c$
2. $c \cdot f$
3. $f \pm g$
4. f^2
5. $f \cdot g$

are all measurable.

Proof.

- ✓

□

Sequencing

Theorem

Let $\{f_n\}$ be a sequence of measurable functions on a common domain D. Then

1. $\sup\{f_1, \ldots, f_n\}$
2. $\inf\{f_1, \ldots, f_n\}$
3. $\sup_{n\to\infty} f_n$
4. $\inf_{n\to\infty} f_n$
5. $\limsup_{n\to\infty} f_n$
6. $\liminf_{n\to\infty} f_n$

are all measurable.

Proof.

1. Set $f = \{f_1, \ldots, f_n\}$. Then $\{f(x) > r\} = \bigcup_{k=1}^{n} \{f_k(x) > r\}$.
3. Set $F = \sup_n f_n$. Then $\{F(x) > r\} = \bigcup_{k=1}^{\infty} \{f_k(x) > r\}$.
5. Set $\Phi = \limsup_n f_n$. Then $\limsup_{n\to\infty} f_n = \inf_n \left[\sup_{k\geq n} f_k\right]$

□

Zeroing

Theorem

If f is measurable and $f = g$ a.e., then g is measurable.

Definition (Converence Almost Everywhere)

A sequence $\{f_n\}$ *converges to f almost everywhere*, written as $f_n \to f$ a.e., iff $\mu\Big(\{x : f_n(x) \not\to f(x)\}\Big) = 0$.

Theorem

Let $f : [a, b] \to \mathbb{R}$. Then f is measurable iff there is a seq. of simple functions $\{\psi_n\}$ converging to f a.e.

A Simple Proof

Proof.

$(\Rightarrow)$ Wolog $f \geq 0$.

1. Define $A_{n,k} = \left\{x \,\middle|\, \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\}$ for $k = 1..(n \cdot 2^n)$ and

$$A_{0,n} = [a,b] - \bigcup_{k=1}^{n2^n} A_{n,k}$$

2. Set $\psi_n(x) = n\chi_{A_{0,n}}(x) + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \cdot \chi_{A_n,k}(x)$
3. Then
   - 3.1 $\psi_1 \leq \psi_2 \leq \cdots$
   - 3.2 If $0 \leq f(x) \leq n$, then $|f - \psi_n| < 2^{-n}$
   - 3.3 $\lim_n \psi = f$ a.e.

$(\Leftarrow)$ ✓

□

Integration

We began by looking at two examples of integration problems.

- The Riemann integral over $[0, 1]$ of a function with infinitely many discontinuities didn't exist even though the points of discontinuity formed a set of measure zero.
  (The points of discontinuity formed a dense set in $[0, 1]$.)
- The limit of a sequence of Riemann integrable functions did not equal the integral of the limit function of the sequence.
  (Each function had area ${}^1\!/_2$, but the limit of the sequence was the zero function.)

We will look at Riemann integration, then Riemann-Stieltjes integration, and last, develop the Lebesgue integral.

There are many other types of integrals: Darboux, Denjoy, Gauge, Perron, etc. See the list given in the "See also" section of *Integrals* on Mathworld.

Riemann Integral

Definition

- A *partition* $\mathcal{P}$ of $[a,b]$ is a finite set of points such that $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a,b]$ w.r.t. $\mathcal{P}$ is
$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$
- The *upper Riemann integral* of f over $[a,b]$ is
$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P}} U(\mathcal{P}, f)$$

Exercise

1. *Define the lower sum* $L(\mathcal{P}, f)$ *and the lower integral* $\underline{\int_a^b} f$.

Definitely a Riemann Integral

Definition

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then f is Riemann integrable and is written as $\int_a^b f(x)\,dx$ and $f \in \mathfrak{R}$ on $[a, b]$.

Proposition

A function f is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \epsilon.$$

Theorem

If f is continuous on $[a, b]$, then $f \in \mathfrak{R}$ on $[a, b]$.

Theorem

If f is bounded on $[a, b]$ with only finitely many points of discontinuity, then $f \in \mathfrak{R}$ on $[a, b]$.

Properties of Riemann Integrals

Proposition

Let f and $g \in \mathfrak{R}$ on $[a,b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf\,dx = c\int_a^b f\,dx$
- $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$
- $f \cdot g \in \mathfrak{R}$
- *if* $f \leq g$, *then* $\int_a^b f\,dx \leq \int_a^b g\,dx$
- $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$
- *Define* $F(x) = \int_a^x f(t)\,dt$. *Then* F *is continuous and, if* f *is continuous at* x_0, *then* $F'(x_0) = f(x_0)$
- *If* $F' = f$ *on* $[a,b]$, *then* $\int_a^b f(x)\,dx = F(b) - F(a)$

Riemann Integrated Exercises

Exercises

1. *If* $\int_a^b |f(x)|\,dx = 0$, *then* $f = 0$.
2. *Show why* $\int_0^1 \chi_{\mathbb{Q}}(x)\,dx$ *does not exist.*
3. *Define*

$$S_n(x) = \sum_{k=1}^{n+1} \left(\frac{k-1}{k} \cdot \chi_{\left[\frac{k-1}{k}, \frac{k}{k+1}\right)}(x) \right) + \frac{n}{n+1} \chi_{\left[\frac{n+1}{n+2}, 1\right]}(x).$$

   3.1 *How many discontinuities does* S_n *have?*
   3.2 *Prove that* $S_n'(x) = 0$ *a.e.*
   3.3 *Calculate* $\int_0^1 S_n(x)\,dx$.
   3.4 *What is* S_∞*?*
   3.5 *Does* $\int_0^1 S_\infty(x)\,dx$ *exist?*

   (See an animated graph of S_N*.)*

Riemann-Stieltjes Integral

Definition

- Let $\alpha(x)$ be a monotonically increasing function on $[a, b]$. Set $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a, b]$ w.r.t. α and $\mathcal{P}$ is

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i$$

- The *upper Riemann-Stieltjes integral* of f over $[a, b]$ w.r.t. α is

$$\overline{\int_a^b} f(x)\, d\alpha(x) = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha)$$

Exercise

1. *Define the lower sum $L(\mathcal{P}, f, \alpha)$ and lower integral $\underline{\int_a^b} f\, d\alpha$.*

Definitely a Riemann-Stieltjes Integral

Definition

If $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, then f is Riemann-Stieltjes integrable and is written as $\int_a^b f(x)\, d\alpha(x)$ and $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Proposition

A function f is Riemann-Stieltjes integrable w.r.t. α on $[a, b]$ iff for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon.$$

Theorem

If f is continuous on $[a, b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Theorem

If f is bounded on $[a, b]$ with only finitely many points of discontinuity and α is continuous at each of f's discontinuities, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Properties of Riemann-Stieltjes Integrals

Proposition

Let f *and* $g \in \mathfrak{R}(\alpha)$ *and in* β *on* $[a, b]$ *and* $c \in \mathbb{R}$. *Then*

- $\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$ *and* $\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha$
- $\int_a^b (f + g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$ *and* $\int_a^b f\, d(\alpha + \beta) = \int_a^b f\, d\alpha + \int_a^b f\, d\beta$
- $f \cdot g \in \mathfrak{R}(\alpha)$
- *if* $f \leq g$, *then* $\int_a^b f\, d\alpha \leq \int_a^b g\, d\alpha$
- $\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f|\, d\alpha$
- *Suppose that* $\alpha' \in \mathfrak{R}$ *and* f *is bounded. Then* $f \in \mathfrak{R}(\alpha)$ *iff* $f\alpha' \in \mathfrak{R}$ *and*
$$\int_a^b f\, d\alpha = \int_a^b f \cdot \alpha'\, dx$$

Riemann-Stieltjes Integrals and Series

Proposition

If f is continuous at $c \in (a, b)$ and $\alpha(x) = r$ for $a \leq x < c$ and $\alpha(x) = s$ for $c < x \leq b$, then

$$\int_a^b f\, d\alpha = f(c)\,(\alpha(c+) - \alpha(c-))$$
$$= f(c)\,(s - r)$$

Proposition

Let $\alpha = \lfloor x \rfloor$, the greatest integer function. If f is continuous on $[0, b]$, then

$$\int_0^b f(x)\, d\lfloor x \rfloor = \sum_{k=1}^{\lfloor b \rfloor} f(k)$$

Riemann-Stieltjes Integrated Exercises

Exercises

1. $\int_0^1 x\,dx^2$
2. $\int_0^{\pi/2} \cos(x)\,d\sin(x)$
3. $\int_0^{5/2} x\,d(x - \lfloor x \rfloor)$
4. $\int_{-1}^1 e^x d|x|$
5. $\int_{-3/2}^{3/2} e^x d\lfloor x \rfloor$
6. $\int_{-1}^1 e^x d\lfloor x \rfloor$
7. *Set H to be the Heaviside function; i.e.,*

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & \textit{otherwise} \end{cases}.$$

Show that, if f is continuous at 0, then

$$\int_{-\infty}^{+\infty} f(x)\,dH(x) = f(0).$$

Lebesgue Integral

We start with simple functions.

Definition

A function has *finite support* if it vanishes outside a finite interval.

Definition

Let ϕ be a measurable simple function with finite support. If $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ is a representation of ϕ, then

$$\int \phi(x)\, dx = \sum_{i=1}^{n} a_i \cdot \mu(A_i)$$

Definition

If E is a measurable set, then $\int_E \phi = \int \phi \cdot \chi_E$.

Integral Linearity

Proposition

If ϕ and ψ are measurable simple functions with finite support and $a, b \in \mathbb{R}$, then $\displaystyle\int (a\phi + b\psi) = a\int \phi + b\int \psi$. Further, if $\phi \le \psi$ a.e., then $\displaystyle\int \phi \le \int \psi$.

Proof (sketch).

I. Let $\phi = \displaystyle\sum^{N} \alpha_i \chi_{A_i}$ and $\psi = \displaystyle\sum^{M} \beta_i \chi_{B_i}$. Then show $a\phi + b\psi$ can be written as $a\phi + b\psi = \displaystyle\sum^{K} (a\alpha_{k_i} + b\beta_{k_j})\chi_{E_k}$ for the properly chosen E_k. Set A_0 and B_0 to be zero sets of ϕ and ψ. (Take $\{E_k : k = 0..K\} = \{A_j \cap B_k : j = 0..N, k = 0..M\}$.)

II. Use the definition to show $\int \psi - \int \phi = \int (\psi - \phi) \ge \int 0 = 0$. □

Steps to the Lebesgue Integral

Proposition

Let f be bounded on $E \in \mathfrak{M}$ with $\mu(E) < \infty$. Then f is measurable iff

$$\inf_{f \leq \psi} \int_E \psi = \sup_{f \geq \phi} \int_E \phi$$

for all simple functions ϕ and ψ.

Proof.

I. Suppose f is bounded by M. Define

$$E_k = \left\{ x : \frac{k-1}{n} M < f(x) \leq \frac{k}{n} M \right\}, \qquad -n \leq k \leq n$$

The E_k are measurable, disjoint, and have union E. Set

$$\psi_n(x) = \frac{M}{n} \sum_{-n}^{n} k \, \chi_{E_k}(x), \qquad \phi_n(x) = \frac{M}{n} \sum_{-n}^{n} (k-1) \, \chi_{E_k}(x)$$

□

SLI (cont)

(proof cont).

Then $\phi_n(x) \le f(x) \le \psi(x)$, and so

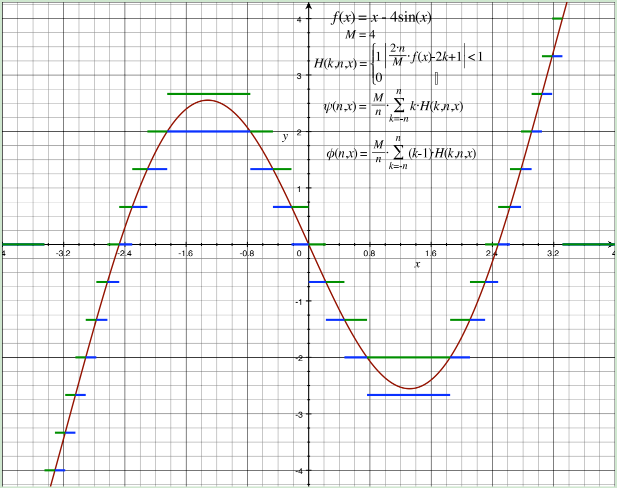
- $\inf \int_E \psi \le \int_E \psi_n = \frac{M}{n} \sum_{k=-n}^{n} k\,\mu(E_k)$
- $\sup \int_E \phi \ge \int_E \phi_n = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\,\mu(E_k)$

Thus $0 \le \inf \int_E \psi - \sup \int_E \phi \le \frac{M}{n}\mu(E)$. Since n is arbitrary, equality holds.

II. Suppose that $\inf \int_E \psi = \sup \int_E \phi$. Choose ϕ_n and ψ_n so that $\phi_n \le f \le \psi_n$ and $\int_E(\psi_n - \phi_n) < \frac{1}{n}$. The functions $\psi^* = \inf \psi_n$ and $\phi^* = \sup \phi_n$ are measurable and $\phi^* \le f \le \psi^*$. The set $\Delta = \{x : \phi^*(x) < \psi^*(x)\}$ has measure 0. Thus $\phi^* = \psi^*$ almost everywhere, so $\phi^* = f$ a.e. Hence f is measurable. $\square$

Example Steps

Example



Defining the Lebesgue Integral

Definition

If f is a bounded measurable function on a measurable set E with $m(E) < \infty$, then

$$\int_E f = \inf_{\psi \geq f} \int_E \psi$$

for all simple functions $\psi \geq f$.

Proposition

Let f be a bounded function defined on $E = [a, b]$. If f is Riemann integrable on $[a, b]$, then f is measurable on $[a, b]$ and

$$\int_E f = \int_a^b f(x)\, dx;$$

the Riemann integral of f equals the Lebesgue integral of f.

Properties of the Lebesgue Integral

Proposition

If f and g are measurable on E, a set of finite measure, then

- $\displaystyle\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
- *if* $f = g$ *a.e., then* $\displaystyle\int_E f = \int_E g$
- *if* $f \leq g$ *a.e., then* $\displaystyle\int_E f \leq \int_E g$
- $\displaystyle\left|\int_E f\right| \leq \int_E |f|$
- *if* $a \leq f \leq b$*, then* $\displaystyle a \cdot \mu(E) \leq \int_E f \leq b \cdot \mu(E)$
- *if* $A \cap B = \emptyset$*, then* $\displaystyle\int_{A \cup B} f = \int_A f + \int_B f$

Lebesgue Integral Examples

Examples

1. Let $T(x) = \begin{Bmatrix} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} T = \int_0^1 T(x)\,dx$.
2. Let $\chi_{\mathbb{Q}}(x) = \begin{Bmatrix} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} \chi_{\mathbb{Q}} \neq \int_0^1 \chi_{\mathbb{Q}}(x)dx$.
3. Define
$$f_n(x) = \sum_{k=1}^{n+1} \left(\frac{k-1}{k} \cdot \chi_{\left[\frac{k-1}{k}, \frac{k}{k+1}\right)}(x) \right) + \frac{n}{n+1} \chi_{\left[\frac{n+1}{n+2}, 1\right]}(x).$$
Then
   3.1 f_n is a step function, hence integrable
   3.2 $f_n'(x) = 0$ a.e.
   3.3 $\frac{1}{4} \leq \int_{[0,1]} f_n = \int_0^1 f_n(x)\,dx < \frac{3}{8}$

Extending the Integral Definition

Definition

Let f be a nonnegative measurable function defined on a measurable set E. Define

$$\int_E f = \sup_{h \le f} \int_E h$$

where h is a bounded measurable function with finite support.

Proposition

If f and g are nonnegative measurable functions, then

- $\displaystyle\int_E c\,f = c\int_E f$ *for* $c > 0$
- $\displaystyle\int_E f + g = \int_E f + \int_E g$
- *If $f \le g$ a.e., then* $\displaystyle\int_E f \le \int_E g$

General Lebesgue's Integral

Definition

Set $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. A measurable function f is integrable over E iff both f^+ and f^- are integrable over E, and then $\displaystyle\int_E f = \int_E f^+ - \int_E f^-$.

Proposition

Let f and g be integrable over E and let $c \in \mathbb{R}$. Then

1. $\displaystyle\int_E cf = c\int_E f$
2. $\displaystyle\int_E f + g = \int_E f + \int_E g$
3. *if $f \leq g$ a.e., then* $\displaystyle\int_E f \leq \int_E g$
4. *if A, B are disjoint m'ble subsets of E,* $\displaystyle\int_{A\cup B} f = \int_A f + \int_B f$

Convergence Theorems

Theorem (Bounded Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging to f with $m(E) < \infty$. If there is a uniform bound M for all f_n, then

$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Proof (sketch).

Let $\epsilon > 0$.

1. f_n converges "almost uniformly;" i.e., $\exists A, N$ s.t. $m(A) < \dfrac{\epsilon}{4M}$ and, for $n > N, x \in E - A \implies |f_n(x) - f(x)| \leq \dfrac{\epsilon}{2\,m(E)}$.
2. $\left|\int_E f_n - \int_E f\right| = \left|\int_E f_n - f\right| \leq \int_E |f_n - f| = \left(\int_{E-A} + \int_A\right) |f_n - f|$
3. $\int_{E-A} |f_n - f| + \int_A |f_n| + |f| \leq \dfrac{\epsilon}{2m(E)} \cdot m(E) + 2M \cdot \dfrac{\epsilon}{4M} = \epsilon$ □

Lebesgue's Dominated Convergence Theorem

Theorem (Dominated Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging a.e. on E with $m(E) < \infty$. If there is an integrable function g on E such that $|f_n| \le g$ then

$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Lemma

Under the conditions of the DCT, set $g_n = \sup_{k \ge n} \{f_n, f_{n+1}, \dots\}$ and $h_n = \inf_{k \ge n} \{f_n, f_{n+1}, \dots\}$. Then g_n and h_n are integrable and $\lim g_n = f = \lim h_n$ a.e.

Proof of DCT (sketch).

- Both g_n and h_n are monotone and converging. Apply MCT.
- $h_n \le f_n \le g_n \implies \int_E h_n \le \int_E f_n \le \int_E g_n$.

□

Increasing the Convergence

Theorem (Fatou's Lemma)

If $\{f_n\}$ is a sequence of measurable functions converging to f a.e. on E, then

$$\int_E \lim_n f_n \leq \liminf_n \int_E f_n$$

Theorem (Monotone Convergence Theorem)

If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging to f, then

$$\int \lim_n f_n = \lim_n \int f_n$$

Corollary (Beppo Levi Theorem (cf.))

If $\{f_n\}$ is a sequence of nonnegative measurable functions, then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Sidebar: Littlewood's Three Principles

John Edensor Littlewood said,

> *The extent of knowledge required is nothing so great as sometimes supposed. There are three principles, roughly expressible in the following terms:*
>
> - *every measurable set is nearly a finite union of intervals;*
> - *every measurable function is nearly continuous;*
> - *every convergent sequence of measurable functions is nearly uniformly convergent.*
>
> *Most of the results of analysis are fairly intuitive applications of these ideas.*

From *Lectures on the Theory of Functions,* Oxford, 1944, p. 26.

Extensions of Convergence

The sequence f_n converges to f ...

Definition (Convergence Almost Everywhere)

almost everywhere if $m(\{x : f_n(x) \nrightarrow f(x)\}) = 0$.

Definition (Convergence Almost Uniformly)

almost uniformly on E if, for any $\epsilon > 0$, there is a set $A \subset E$ with $m(A) < \epsilon$ so that f_n converges uniformly on $E - A$.

Definition (Convergence in Measure)

in measure if, for any $\epsilon > 0$, $\lim_{n\to\infty} m\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) = 0$.

Definition (Convergence in Mean (of order $p > 1$))

in mean if $\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \left[\int_E |f - f_n|^p\right]^{1/p} = 0$

Integrated Exercises

Exercises

1. *Prove: If f is integrable on E, then $|f|$ is integrable on E.*
2. *Prove: If f is integrable over E, then* $\left|\int_E f\right| \le \int_E |f|$.
3. *True or False: If $|f|$ is integrable over E, then f is integrable over E.*
4. *Let f be integrable over E. For any $\epsilon > 0$, there is a simple (resp. step) function ϕ (resp. ψ) such that* $\int_E |f - \phi| < \epsilon$.
5. *For $n = k + 2^\nu, 0 \le k < 2^\nu$, define* $f_n = \chi_{[k2^{-\nu},(k+1)2^{-\nu}]}$.
   5.1 *Show that f_n does not converge for any $x \in [0,1]$.*
   5.2 *Show that f_n does not converge a.e. on $[0,1]$.*
   5.3 *Show that f_n does not converge almost uniformly on $[0,1]$.*
   5.4 *Show that $f_n \to 0$ in measure.*
   5.5 *Show that $f_n \to 0$ in mean (of order 2).*

References

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