MAT 5620	Final Exam	NAME:________________
FALL '11		ASU EMAIL:________________

Work quickly and carefully, following directions closely. Answer all questions completely.

§I. PROBLEMS.

1. Prove or disprove: If $\vec{r}\colon[a,b]\to\mathbb{R}^n$ is integrable, then

$$\left\| \int_a^b \vec{r}(t)\,dt \right\| \le \int_a^b \|\vec{r}(t)\|\ dt$$

2. Let $A = \left\{ \left(\frac{1}{n}\cos(\frac{1}{n}), \frac{1}{n^2}\sin(\frac{1}{n^2})\right) : n \in \mathbb{N} \right\} \subset \mathbb{R}^2$
   (a) Is A a connected set? If so, prove it; if not, why not?
   (b) Is A is separated from $B = \{\,\vec{0}\,\}$?

3. Prove: Let f and g be differentiable functions at $\vec{a}$. Then $\nabla(f+g) = \nabla f + \nabla g$.

4. Prove: Let f be continuous and integrable on a closed and bounded region R. Then

$$\operatorname{area}(R)\cdot\min_{x\in R} f(x) \le \iint_R f\,dA \le \operatorname{area}(R)\cdot\max_{x\in R} f(x)$$

5. Let C be a smooth rectifiable curve, and let f be continuous everywhere. Then

$$\left| \int_C f\,ds \right| \le \operatorname{length}(C)\cdot\max_{\vec{x}\in C} |f(\vec{x})|$$

6. Show that any subset of set of measure zero has measure zero.

7. Compute
   (a) $\displaystyle\int_{[0,1]} \chi_{\mathbb{Q}}\,d\mu$
   (b) $\displaystyle\int_{[0,1]} T\,d\mu$ where $T(x)$ is Thomae's function.

§II. EXTRA CREDIT.

1. Fix a real number $a \ge 0$. Define a sequence $f_n\colon[0,\infty)\to\mathbb{R}$ by $f_n(x) = \dfrac{x^n}{1+x^n}$. Determine $\displaystyle\lim_{n\to\infty}\int_0^{a+1} f_n(x)\,dx$.