1 Uniform

Definition 1 (Uniform Continuity)**.** *A function $f : D \to \mathbb{R}$ is* uniformly continuous on D *iff for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $\vec{x}_1, \vec{x}_2 \in D$, if $\|\vec{x}_1 - \vec{x}_2\| < \delta$, then $|f(\vec{x}_1) - f(\vec{x}_2)| < \varepsilon$.*

Theorem 1. *If f is continuous on D, and D is closed & bounded (compact), then*

1. *f is bounded,*
2. *f attains extreme values (max and min),*
3. *f is uniformly continuous on D.*

Proof (Homework). (Keys)

1. Hint: Assume not, then look at $f^{-1}(a_n)$ where $a_n \to \infty$.
2. *Bolzano-Weierstrass in action.*
3. Hint: Assume not. Create sequences $\vec{x}_n$, $\vec{y}_n$ that converge to $\vec{a}$, but have $|f(\vec{x}_n) - f(\vec{y}_n)| > \varepsilon$. Cont gives a contradiction.

□

2 Fun with Functions

Problem 1 (Functions)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let A and B be subsets of the domain and range of f, respectively. Then*

$$f(A) = \{y \in \mathbb{R} \mid f(a) = y \text{ for some } a \in A\} \subseteq \operatorname{range}(f)$$
$$f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) = b \text{ for some } b \in B\} \subseteq \operatorname{dom}(f)$$

Give an example justifying your answer.

1. **T** or **F**: $A \subseteq f^{-1}\big(f(A)\big)$
2. **T** or **F**: $A = f^{-1}\big(f(A)\big)$
3. **T** or **F**: $A \supseteq f^{-1}\big(f(A)\big)$ *or* $f^{-1}\big(f(A)\big) \subseteq A$
4. **T** or **F**: $B \subseteq f\big(f^{-1}(B)\big)$
5. **T** or **F**: $B = f\big(f^{-1}(B)\big)$
6. **T** or **F**: $B \supseteq f\big(f^{-1}(B)\big)$ *or* $f\big(f^{-1}(B)\big) \subseteq B$