Vector Calculus

Vector Space Axioms

A set $\mathcal{V} = \{\vec{v}\}$ with addition $+$ and scalar multiplication $\cdot$ with scalars from a field F is a *vector space over F* when

1. $\langle \mathcal{V}, + \rangle$ is an Abelian group.
2.
   - scalar multiplication distributes over vector addition
   - scalar addition distributes over scalar multiplication
   - multiplication of scalars 'associates' with scalar multiplication

Recall:

- The *norm* (magnitude) of a vector $\vec{u}$ is $\|\vec{u}\| = \sqrt{\sum u_i^2}$
- The *direction vector* of $\vec{u}$ is $(1/\|\vec{u}\|) \cdot \vec{u}$

Definition (Dot Product in $\mathbb{R}^n$ over $\mathbb{R}$)

Dot Product $\quad \vec{u} \cdot \vec{v} = \sum u_i \cdot v_i = \|\vec{u}\| \, \|\vec{v}\| \, \cos(\angle \overline{uv})$

Dot Product

Proposition (Dot Product Properties)

Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. Then

1. $\angle \overline{uv} = \cos^{-1}\left[\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right]$ *angle between vectors*
2. $|\vec{u} \cdot \vec{v}| \le \|\vec{u}\| \, \|\vec{v}\|$ *Cauchy-Bunyakovsky-Schwarz inequality*
3. $\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$ *Triangle inequality; (cf. Minkowski's inequality)*
4. $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \, \vec{v}$ *(orthogonal) projection of $\vec{u}$ onto $\vec{v}$*

Cross Product

Definition

- Let $\vec{u}$ and $\vec{v} \in \mathbb{R}^3$; set e_1, e_2, e_3 to be std basis vectors. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- Let $\vec{u_1}$ to $\vec{u}_{n-1} \in \mathbb{R}^n$, $n \geq 3$; let $\{e_n\} = \{\text{std basis vectors}\}$. Then

$$\times(\vec{u_1}, \dots, \vec{u}_{n-1}) = \begin{vmatrix} e_1 & e_2 & \dots & e_n \\ u_{1,1} & u_{1,2} & \dots & u_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1,1} & u_{n-1,2} & \dots & u_{n-1,n} \end{vmatrix}$$

Cross Product Properties

Proposition (Cross Product Properties in $\mathbb{R}^3$)

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be in $\mathbb{R}^3$. Then

1. $\angle\overline{uv} = \sin^{-1}\left[\dfrac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\|\,\|\vec{v}\|}\right]$ *angle between vectors*
2. $\|\vec{u} \times \vec{v}\| \leq \|\vec{u}\|\,\|\vec{v}\|$
3. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ *area of* $[\vec{u}, \vec{v}] = \|\vec{u} \times \vec{v}\|$
4. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{v} \cdot (\vec{w} \times \vec{u})$
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$; *volume of* $[\vec{u}, \vec{v}, \vec{w}] = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

Cross Product Anime References

Parametric Equations

Definition (Parametrization)

Suppose $f\colon D \to \mathbb{R}$, $g\colon D \to \mathbb{R}$, and $h\colon D \to \mathbb{R}$. Then

$$\gamma(t) = (f(t), g(t), h(t))$$

for $t \in D$ is a *curve (spacecurve)* in $\mathbb{R}^3$. The fcns f, g, and h are *parametric equations* for γ, or a *parametrization of* γ.

Examples

1. The line segment L from $\vec{u}$ to $\vec{w}$ can be parametrized as

$$L(t) = \vec{u} + (\vec{w} - \vec{u}) \cdot t, \qquad t \in [0, 1]$$

2. Γ given by `f:=t->⟨cos(t),sin(t)*cos(t),t*(1-t)⟩` for $t \in [0, 3\pi]$.

```
animate(spacecurve,[f(t),t=0..3*Pi*k,
thickness=2],k=0..1,axes=frame,color=black,frames=30)
```


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Continuous Spacecurves

Definition

Let $\mathcal{I} = [a, b] \subseteq \mathbb{R}$. A curve γ is

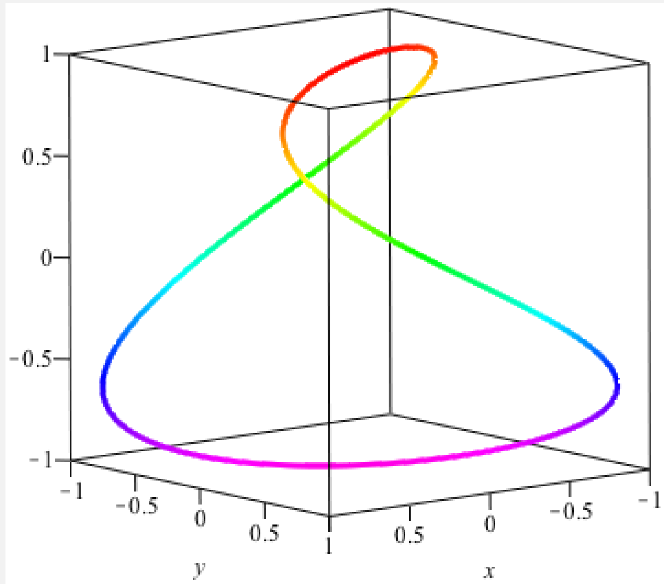
- *continuous (on $\mathcal{I}$)* if γ can be parametrized with components that are continuous on $\mathcal{I}$.
- *smooth (on $\mathcal{I}$)* if γ's parametric components are continuously differentiable on $\mathcal{I}$, and ${f'}^2 + {g'}^2 + {h'}^2 > 0$ for all $t \in (a, b)$.
- *piecewise smooth (on $\mathcal{I}$)* if $[a, b]$ can be partitioned into a finite number of subintervals on which γ is smooth.

Note: Smooth $\equiv$ a particle moving parametrically along the curve doesn't change direction abruptly, stop mid-curve, or reverse.

Theorem

If $\gamma(t) = (f(t), g(t))$ is smooth on $[a, b]$, then tangent slope at $P_0 = (x, y)$ is given by $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$ when $\dfrac{dx}{dt} \neq 0$.

A Smooth Closed Curve



$\Gamma(t) = (\sin(2t), \sin(t), \cos(t))$ for $t \in [0, 2\pi]$

$$\Gamma(0) = \Gamma(2\pi)$$

Lines in $\mathbb{R}^3$

Theorem (The Line Forms Here Thm)

A line ℓ passing through $P_0 = (x_0, y_0, z_0)$, parallel to $\vec{u} = (a, b, c) \neq \vec{0}$ has

vector form: $\ell(t) = P_0 + t\,\vec{u}$, $t \in \mathbb{R}$

parametric form: $\ell(t) = (x_0 + at,\ y_0 + bt,\ z_0 + ct)$, $t \in \mathbb{R}$

symmetric form: $\dfrac{x(t) - x_0}{a} = \dfrac{y(t) - y_0}{b} = \dfrac{z(t) - z_0}{c}$

Consider...

Let $P_0 = (1, 2, 4)$ and direction $\vec{u} = (1, 2, -1)$.

1. $\ell_1(t) = (1 + t,\ 2 + 2t,\ 4 - t)$ $\qquad \vec{u} = (1, 2, -1)$
2. $\ell_2(s) = \left(1 + \frac{1}{\sqrt{6}}\,s,\ 2 + \frac{2}{\sqrt{6}}\,s,\ 4 - \frac{1}{\sqrt{6}}\,s\right)$ $\qquad \vec{w} = \frac{1}{\sqrt{6}}(1, 2, -1)$

Planes in $\mathbb{R}^3$

Theorem (The Plane, the Plane)

A plane P passing through $P_0 = (x_0, y_0, z_0)$, normal to $\vec{u} = (a, b, c) \neq \vec{0}$ is $P = \{\vec{X}\}$ s.t.

vector form: $\vec{u} \cdot \left(\vec{X} - P_0\right) = 0$

parametric form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

A plane P passing through $P_0 = (x_0, y_0, z_0)$, containing two vectors $\vec{u}$ and $\vec{w}$ is $P = \{\vec{X}\}$ s.t.

cross product form: $(\vec{u} \times \vec{w}) \cdot \left(\vec{X} - P_0\right) = 0$

Problem

1. *Find a plane containing the three points* $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

Quadric Surfaces

Standard Forms of Quadric Surfaces

sphere: $x^2 + y^2 + z^2 = r^2$

ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

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elliptic paraboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z = 0$

hyperbolic paraboloid: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + z = 0$

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elliptic cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - z^2 = 0$

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hyperboloid of 1 sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = +1$

hyperboloid of 2 sheets: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$

Quadric Surfaces Reformed

Almost Standard Forms of Quadric Surfaces

sphere:	$\rho x^2 + \rho y^2 + \rho z^2 = 1$
ellipsoid:	$\alpha x^2 + \beta y^2 + \gamma z^2 = 1$
elliptic paraboloid:	$\alpha x^2 + \beta y^2 - z = 0$
hyperbolic paraboloid:	$\alpha x^2 - \beta y^2 + z = 0$
elliptic cone:	$\alpha x^2 + \beta y^2 - z^2 = 0$
hyperboloid of 1 sheet:	$\alpha x^2 + \beta y^2 - \gamma z^2 = +1$
hyperboloid of 2 sheets:	$\alpha x^2 + \beta y^2 - \gamma z^2 = -1$

Vector-Valued Functions

Notation

The *standard basis vectors* in $\mathbb{R}^3$ are

$$\langle 1,0,0\rangle = e_1 = \mathbf{i}, \qquad \langle 0,1,0\rangle = e_2 = \mathbf{j}, \qquad \langle 0,0,1\rangle = e_3 = \mathbf{k}$$

If f, g, $h : D \to \mathbb{R}$ are real functions, then $\vec{r} : D \to \mathbb{R}^3$ given by

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is a *vector-valued function* with components f, g, and h.

Definition

Let $\vec{r} : D \to \mathbb{R}^3$ have components f, g, and h, and let t_0 be an accumulation point of D. Then

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L} = L_f\mathbf{i} + L_g\mathbf{j} + L_h\mathbf{k}$$

iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t \in D)$ if $0 < |t - t_0| < \delta$, then $\|\vec{r}(t) - \vec{L}\| < \epsilon$.

Vector-Valued Function Limits

Theorem (Limits Work)

$$\lim_{t\to t_0} \vec{r}(t) = L_f\mathbf{i} + L_g\mathbf{j} + L_h\mathbf{k}$$

$$\iff$$

$$\lim_{t\to t_0} f(t) = L_f \ \wedge \ \lim_{t\to t_0} g(t) = L_g \ \wedge \ \lim_{t\to t_0} h(t) = L_h$$

Proof (key inequality).

$$|a| \underset{(\Leftarrow)}{\leq} \sqrt{a^2+b^2+c^2} = \|(a,b,c)\| \underset{(\Rightarrow)}{\leq} |a| + |b| + |c|$$

$\square$

Algebra of Vector-Valued Function Limits

Theorem (Algebra of Vector-Valued Limits)

Suppose $\vec{u}, \vec{w} : D \to \mathbb{R}^n$, $k : D \to \mathbb{R}$, $c \in \mathbb{R}$, *and* $t_0 \in D'$. *Then*

$$\lim_{t\to t_0} [\vec{u} \pm \vec{w}] = \left[\lim_{t\to t_0} \vec{u}\right] \pm \left[\lim_{t\to t_0} \vec{w}\right] \tag{1}$$

$$\lim_{t\to t_0} [c\vec{u}] = c\left[\lim_{t\to t_0} \vec{u}\right] \tag{2}$$

$$\lim_{t\to t_0} [k\vec{u}] = \left[\lim_{t\to t_0} k\right]\left[\lim_{t\to t_0} \vec{u}\right] \tag{3}$$

$$\lim_{t\to t_0} [\vec{u} \cdot \vec{w}] = \left[\lim_{t\to t_0} \vec{u}\right] \cdot \left[\lim_{t\to t_0} \vec{w}\right] \tag{4}$$

$$\lim_{t\to t_0} [\vec{u} \times \vec{w}] = \left[\lim_{t\to t_0} \vec{u}\right] \times \left[\lim_{t\to t_0} \vec{w}\right] \tag{5}$$

Continuity of Vector-Valued Functions

Definition (Continuity)

A function $\vec{r}(t)$ is *continuous* at $t_0 \in D$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t \in D)$ if $|t - t_0| < \delta$, then $\|\vec{r}(t) - \vec{r}(t_0)\| < \epsilon$.

Proposition

1. *A function $\vec{r}(t)$ is continuous at an accumulation point $t_0 \in D$ iff*
$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$$
2. *A function $\vec{r}(t)$ is uniformly continuous on $E \subseteq D$ iff $(\forall \epsilon > 0)$ $(\exists \delta > 0)$ s.t. $(\forall t_1, t_2 \in E)$ if $|t_1 - t_2| < \delta$, then $\|\vec{r}(t_1) - \vec{r}(t_2)\| < \epsilon$.*
3. *If a function $\vec{r}(t)$ is continuous on a closed and bounded set E, then $\vec{r}$ is uniformly continuous on E.*

Differentiability of Vector-Valued Functions

Definition (Differentiable)

A function $\vec{r}(t)$ is *differentiable* at $t_0 \in D$ iff the limit
$$\vec{r}\,'(t) = \lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} = \lim_{h \to 0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}$$
exists and is finite.

Proposition

If f, g, and h are the components of $\vec{r}$, then $\vec{r}$ is differentiable iff f, g, and h are differentiable, whence
$$\vec{r}\,'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example

1. Find $\vec{r}\,'$ for the line through $P_0 = (1, 2, 4)$ parallel to $\vec{u} = (1, 2, -1)$.

Algebra of Vector-Valued Derivatives

Theorem (Algebra of Derivatives)

Suppose $\vec{u}, \vec{w} : D \to \mathbb{R}^n$ *&* $k : D \to \mathbb{R}$ *are all differentiable, and* $c \in \mathbb{R}$*. Then*

$$[\vec{u} \pm \vec{w}]' = [\vec{u}\,'] \pm [\vec{w}\,'] \tag{6}$$

$$[c\,\vec{u}]' = c\,[\vec{u}\,'] \tag{7}$$

$$[k\,\vec{u}]' = [k']\ \vec{u} + k\ [\vec{u}\,'] \tag{8}$$

$$[\vec{u} \cdot \vec{w}]' = [\vec{u}\,'] \cdot \vec{w} + \vec{u} \cdot [\vec{w}\,'] \tag{9}$$

$$[\vec{u} \times \vec{w}]' = [\vec{u}\,'] \times \vec{w} + \vec{u} \times [\vec{w}\,'] \tag{10}$$

$$\|\vec{u}\|' = \frac{\vec{u} \cdot [\vec{u}\,']}{\|\vec{u}\|} \tag{11}$$

$$[\vec{u} \circ k]' = [\vec{u}\,' \circ k] * k' \tag{12}$$

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Derivative Props

Properties

Suppose $\vec{r}(t)$ is a twice differentiable vector function.

1. $\vec{V}(t) = \vec{r}\,'(t)$ is
   - the *tangent vector* of $\vec{r}$
   - the *velocity vector* of $\vec{r}$

   and $S(t) = \|\vec{r}\,'(t)\|$ gives the *speed* of $\vec{r}(t)$
2. $\vec{A}(t) = \vec{V}\,'(t) = \vec{r}\,''(t)$ is
   - the *acceleration vector* of $\vec{r}$

Example

Find the velocity & acceleration and the speed for the function

1. $\vec{r}(t) = \langle 2\cos(t), 3\sin(t), z_0 \rangle$.
2. $\vec{\rho}(t) = \langle \cos(t) \cdot (1 + \cos(t)), 2\sin(t) \cdot (1 + t), t \rangle$.[1]

[1] `spacecurve(f(t),t=0..6*Pi,numpoints=101,thickness=3,axes=normal)`

Example 9.6.9

Example (9.6.9, pg 410)

Consider $\vec{u}, \vec{v}, \vec{w} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\vec{u} = \langle t, t^2 \rangle, \vec{v} = \langle t^3, t^6 \rangle, \text{ and } \vec{w} = \begin{cases} \langle t, t^2 \rangle & \text{if } t \le 0 \\ \langle t^3, t^6 \rangle & \text{if } t > 0 \end{cases}$$

All 3 functions are continuous, all trace the parabola $y = x^2$, and all are $\vec{0}$ at $t = 0$.

1. $\vec{u}$ is differentiable at $t = 0$ with tangent vector $\vec{u}\,'(0) = \langle 1, 0 \rangle$ and tangent line $y = 0$.
2. $\vec{v}$ is differentiable at $t = 0$ with tangent vector $\vec{v}\,'(0) = \langle 0, 0 \rangle$, but has *no* tangent line $\vec{0}$.
3. $\vec{w}$ is *not* differentiable at $t = 0$ and has no tangent line at $\vec{0}$.

See Maple demo

Circles

Proposition

Let $\vec{r}$ be a differentiable vector function of t. Then $\|\vec{r}(t)\|$ is constant iff $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$; i.e. $\vec{r}$ and $\vec{r}\,'$ are orthogonal.

Proof.

$$\|\vec{r}(t)\| \text{ is constant } \iff \vec{r}(t) \cdot \vec{r}(t) = c \iff \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

□

Definition

Unit tangent vector: $\vec{T}(t) = \vec{r}\,'(t)/\|\vec{r}\,'(t)\|$

Unit normal vector: $\vec{N}(t) = \vec{T}'(t)/\|\vec{T}'(t)\|$

$\vec{V} = \vec{r}\,'$ and $v = \|\vec{V}\|$. Then $\vec{A} = \vec{V}' = v\,\vec{T}' + v'\,\vec{T}$. Since $\vec{T}' \perp \vec{T}$, then $\vec{A}_{\vec{N}} = v\,\vec{T}'$ and $\vec{A}_{\vec{T}} = v'\,\vec{T}$ forms an orthogonal decomp of $\vec{A}$

Þ^e Cræft

Project

Using

$$\vec{r}'' = \vec{A} = v\,\vec{T}' + v'\,\vec{T} \tag{13}$$

$$\vec{A} = \vec{A}_{\vec{N}} + \vec{A}_{\vec{T}} \tag{14}$$

1. *Compute* $\vec{A} \cdot \vec{T}$*?*
2. *What vector is* $(\vec{A} \cdot \vec{T})\,\vec{T}$*?*
3. *Compute* $\vec{A} - \left(\vec{A} \cdot \vec{T}\right)\vec{T}$*?*
4. *Apply this idea to* $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$*. What are* $\vec{A}$*'s orthognal components?*

Int

Definition

$$\int_a^b \vec{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

iff the integrals exist. I.e., $\int_a^b \langle f_i \rangle(t)\,dt = \left\langle \int_a^b f_i(t)dt \right\rangle$.

Theorem (FToC)

Suppose $\vec{r}(t)$ *is integrable on* $[a,b]$ *and* $\vec{R}(t)$ *is an antiderivative (or primitive) for* $\vec{r}$*. Then*

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

Theorem

Suppose $\vec{r}(t)$ *is integrable on* $[a,b]$*. Then*

$$\left\|\int_a^b \vec{r}(t)\,dt\right\| \leq \int_a^b \|\vec{r}(t)\|\,dt$$

Arclength

Definition (Arclength)

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is

$$L(\gamma) = \sup \{L_Q \mid Q \text{ partitions } [a, b]\}$$

where $L_Q = \sum_k \left\|\gamma(t_k) - \gamma(t_{k-1})\right\|$ for $t_k \in Q$.

Proposition

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is $L(\gamma) = \lim_{|Q|\to 0} L_Q$ where $|Q|$ is the norm of the partition.

Theorem (Useful Arclength Theorem)

Let $\gamma(t) = \vec{r}(t)$ be a smooth curve on $[a, b]$. The length of γ on $[a, b]$ is

$$L(\gamma) = \int_a^b \sqrt{\sum_k (f_k')^2}\, dt = \int_a^b \left\|\vec{r}\,'(t)\right\|\, dt$$

Proof

Proof (UAT).

I. Let Q be a partition. Fix k. Whereupon

$$\sqrt{\sum_j \left[f_j(t_k)) - f_j(t_{k-1})\right]^2} = \|\vec{r}(t_k) - \vec{r}(t_{k-1})\| = \left\|\int_{t_{k-1}}^{t_k} \vec{r}\,'(t)\, dt\right\|$$

Since $\left\|\int \vec{r}\,' dt\right\| \le \int \|\vec{r}\,'\|\, dt$, then $L(\gamma) \le \int_a^b \|\vec{r}\,'(t)\|\, dt$.

II. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $\|\vec{r}(s) - \vec{r}(t)\| < \varepsilon$ for $|s - t| < \delta$. Choose $|Q| < \delta$.

1. $\displaystyle\int_{t_k}^{t_{k+1}} \left\|\vec{r}\,'(t)\right\| dt \le \int_{t_k}^{t_{k+1}} \left\|\vec{r}\,'(t_{k+1})\right\| + \varepsilon\, dt = \int_{t_k}^{t_{k+1}} \left\|\vec{r}\,'(t_{k+1})\right\|\, dt + \varepsilon\, \Delta t_k$
2. $\displaystyle\le \left\|\int_{t_k}^{t_{k+1}} \vec{r}\,'(t) dt\right\| + \left\|\int_{t_k}^{t_{k+1}} \left[\vec{r}\,'(t_{k+1}) - \vec{r}\,'(t)\right] dt\right\| + \varepsilon\, \Delta t_k$
3. $\displaystyle\le \|\vec{r}(t_{k+1}) - \vec{r}(t_k)\| + 2\varepsilon\, \Delta t_k \implies \int_a^b \left\|\vec{r}\,'(t)\right\| dt \le L_Q + 2\varepsilon(b - a)$

Hence $\displaystyle\int_a^b \left\|\vec{r}\,'(t)\right\| dt \le L(\gamma)$. □

Rectified

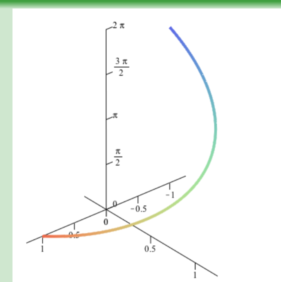
Definition (Recifiable Curve)

A curve γ is *rectifiable* iff $L(\gamma)$ is finite.

Examples (Curves[2])

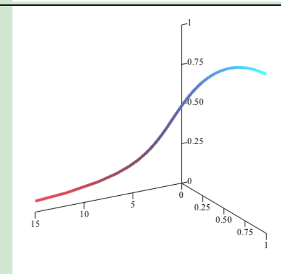
I. Let $\gamma(t) = \langle \cos(\pi t), \sin(\pi t), \sqrt{3}\,\pi t\rangle$ on $[0, 1]$.

1. $L(\gamma) = \displaystyle\int_0^1 \|\gamma'(t)\|\, dt$
2. $= \displaystyle\int_0^1 \left\| \pi\langle -\sin(\pi t), \cos(\pi t), \sqrt{3}\,\rangle \right\| dt = 2\pi$



II. Let $\psi(t) = \langle \tan(t), 1 - \sin(t), \cos(t)\rangle$ on $[0, \pi/2]$.

1. $L(\psi) = \displaystyle\int_0^1 \|\psi'(t)\|\, dt$
2. $= \displaystyle\int_0^1 \left\|\langle \sec^2(t), -\cos(t), -\sin(t)\rangle\right\| dt = \infty$



[2] Maple worksheet

Interlude

Theorem (Most Useful Norm-Integral Estimate)

Let $\vec{r}(t)$ be Riemann integrable on $[a, b]$. Then $\|\vec{r}(t)\|$ is integrable and

$$\left\| \int_a^b \vec{r}(t)\, dt \right\| \leq \int_a^b \|\vec{r}(t)\|\, dt$$

Proof.

I. $\|\vec{r}(t)\|$ is integrable: ✓

II. (in $\mathbb{R}^2$).
$$\left\| \int_a^b \vec{r}(t)\, dt \right\| = \sqrt{\left(\int_a^b f\right)^2 + \left(\int_a^b g\right)^2}$$
$$\leq \sqrt{\int_a^b (f^2) + \int_a^b (g^2)} = \sqrt{\int_a^b (f^2 + g^2)}$$
$$\leq \int_a^b \sqrt{f^2 + g^2} = \int_a^b \|\vec{r}(t)\|\, dt.$$
□

Reparametrize

Definition

Two parametrizations γ_1 on $[a, b]$ and γ_2 on $[c, d]$ of a curve are *equivalent* iff there is a continuously differentiable bijection $u\colon [c, d] \to [a, b]$ such that $u(c) = a$, $u(d) = b$, and $\gamma_2 = \gamma_1 \circ u$.

Theorem

Suppose γ_1 and γ_2 are equivalent smooth parametrizations of a curve. Then $L(\gamma_1) = L(\gamma_2)$.

Proof.

Let u be the equivalence bijection for γ_1 and γ_2. Then

$$L(\gamma_2) = \int_c^d \|\gamma_2'(t)\| dt = \int_c^d \|\gamma_1'(u(t)) \cdot u'(t)\| dt$$

$$= \int_c^d \|\gamma_1'(u(t))\| \cdot u'(t)\, dt = \int_a^b \|\gamma_1(s)\| ds = L(\gamma_1)$$

□

Parametrization by Arclength

Definition (Arclength Parameter)

Set $\ell(t) = \int_a^t \|\vec{r}\,'(\tau)\| d\tau$. Then ℓ is continuous, differentiable, a bijection, and increasing $\Rightarrow$ it has an inverse $\ell^{-1}\colon [0, L(\gamma)] \to [a, b]$. So $\gamma \circ \ell^{-1}\colon [0, L(\gamma)] \to \mathbb{R}^n$ is the *arclength parametrization* of γ.

Example

Let $\vec{r}(t) = \langle \cos(t), \sin(t), t/3 \rangle$ on $[-4\pi, 4\pi]$.

1. Whence $\|\vec{r}\,'(t)\| = \|\langle -\sin(t), \cos(t), 1/3 \rangle\| = \sqrt{10}/3$.
2. Hence $\ell(t) = \int_{-4\pi}^t \sqrt{10}/3\, dt = \sqrt{10}/3 \cdot (t + 4\pi)$.
3. Fortuitously, ℓ is algebraically invertible (*usually not true!*) and $\ell^{-1}(s) = (3/\sqrt{10})s - 4\pi$.
4. Whereupon the arc length parametrized form of γ is

$$\gamma(s) = \left\langle \cos\left(\frac{3}{\sqrt{10}}\, s\right), \sin\left(\frac{3}{\sqrt{10}}\, s\right), \frac{1}{\sqrt{10}}\, s - \frac{4}{3}\,\pi \right\rangle \quad \text{on } \left[0, \frac{8\sqrt{10}}{3}\,\pi\right]$$

Maple