T/F Summary

Prove or Disprove and Salvage

1. If $\vec{u} \times \vec{v} = \vec{0}$, then $\vec{u} = \vec{0}$, or $\vec{v} = \vec{0}$, or both.
2. $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}) = [\vec{u}, \vec{v}, \vec{w}]$.
3. The length of the function f on the interval $[a, b]$ is given by $\int_a^b \sqrt{[f'(x)]^2 + 1}\, dx$, provided that this integral is finite.
4. If $\vec{u}$ is a vector, then $\vec{u} = (\vec{u} \cdot \mathbf{i})\mathbf{i} + (\vec{u} \cdot \mathbf{j})\mathbf{j} + (\vec{u} \cdot \mathbf{k})\mathbf{k}$.
5. If $\gamma : [0, 1] \to \mathbb{R}^2$ is defined by
$$\gamma(t) = \begin{cases} \langle t, t^2 \sin(1/t) \rangle & t \in [0, 1) \\ \langle 0, 0 \rangle & t = 0 \end{cases}$$
then γ is rectifiable.
6. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal, then $\vec{u} \times (\vec{v} \times \vec{w}) = \vec{0}$.
7. If $\vec{u}$ and $\vec{v}$ are Riemann integrable vector-valued functions on $[a, b]$, then $\int_a^b [\vec{u} \cdot \vec{v}]\, dt = \left[\int_a^b \vec{u}\, dt\right] \cdot \left[\int_a^b \vec{v}\, dt\right]$.

Interlude: Inner Products

Definition (Inner Product)

Suppose that $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in a vector space V over the field F, and that $c \in F$ is a scalar. An inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to F$ such that

1. $\langle \vec{u}, \vec{w} \rangle = \langle \vec{w}, \vec{u} \rangle$ — *commutative*
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ — *additivity* (*bi-linear*)
3. $\langle c\,\vec{u}, \vec{v} \rangle = c\,\langle \vec{u}, \vec{v} \rangle$ — *scalar homogeneity* (*bi-linear*)
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$ (*positive definite*)
5. $\langle \vec{u}, \vec{u} \rangle = 0$ iff $\vec{u} = \vec{0}$ (*positive definite*)

Examples

1. The usual dot product on $\mathbb{R}^3$.
2. For $p(x) = \sum^n a_j x^j$ and $q(x) = \sum^n b_j x^j \in \mathbb{P}^n$, set $\langle p, q \rangle = \sum^n a_i b_i$.

Interlude: Orthogonality

Proposition

Suppose that $f(x), g(x) : [a, b] \to \mathbb{R}$ are (piecewise) continuous functions. Then

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$$

is an inner product on the vector space of (piecewise) continuous functions on $[a, b]$

Definition (Orthogonal Vectors)

Suppose that $\vec{u}$ and $\vec{w}$ are vectors in a vector space V over the field F. Then $\vec{u}$ is orthogonal to $\vec{w}$ iff $\langle \vec{u}, \vec{w} \rangle = 0$.

Example (Orthogonal Functions)

1. $\langle \sin, \cos \rangle = \int_{-\pi}^{\pi} \sin(\theta)\cos(\theta)\,d\theta = 0 \implies$ sine $\perp$ cosine on $[-\pi, \pi]$

Interlude: Orthogonal Polynomials

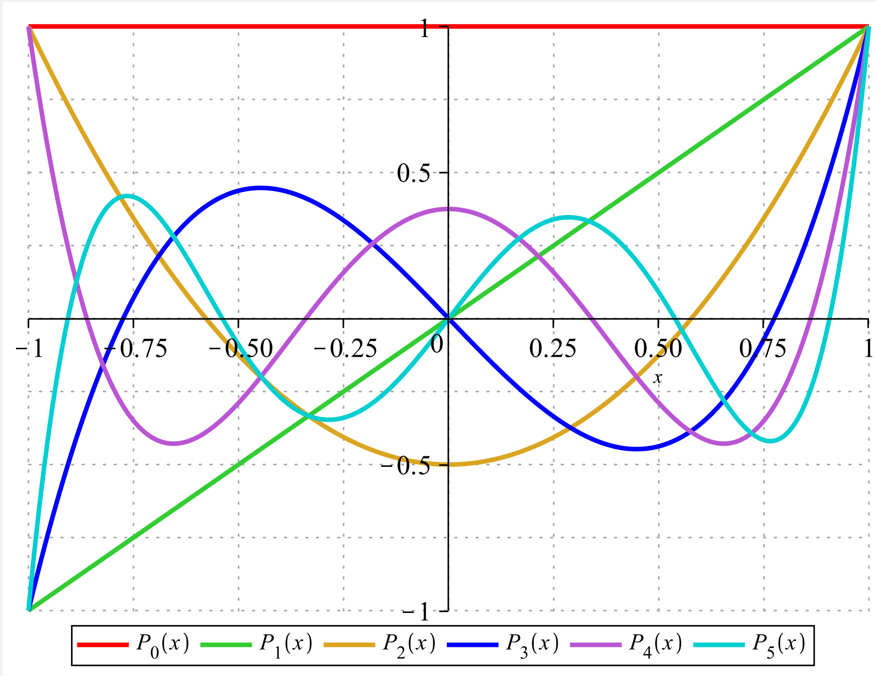
Example (The Legendre Polynomials)

The Legendre polynomials are orthogonal on $[-1, 1]$ wrt $\langle f, g \rangle = \int_{-1}^{1} f g\,dx$. Two formulas for the Legendre polynomials P_n are

1. Rodrigues' formula: $\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left[\left(x^2 - 1\right)^n\right]$.
2. recurrence relation: $(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - nP_{n-1}(x)$.

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \tfrac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \tfrac{1}{2}\left(5x^3 - 3x\right) \\
P_4(x) &= \tfrac{1}{8}\left(35x^4 - 30x^2 + 3\right) \\
P_5(x) &= \tfrac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \\
P_6(x) &= \tfrac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right) \\
P_7(x) &= \tfrac{1}{16}\left(428x^7 - 693x^5 + 315x^3 - 35x\right)
\end{aligned}
$$

Interlude: Legendre Polynomials' Graphs



Maple

Interlude: Expansions in Legendre Polynomials

Proposition (Orthonormalized Legendre Polynomials)

Let $p_n(x) = \sqrt{\frac{2n+1}{2}} \cdot P_n(x)$. *Then* $\langle p_n, p_m \rangle = \delta_{m,n}$.

Theorem

Let f *be integrable on* $[-1,1]$, *and set* $a_n = \langle f, p_n \rangle$. *Then*

$$f_n(x) = \sum_{k=0}^{n} a_n\, p_n(x) \underset{n}{\longrightarrow} f(x)$$

Example

For $f(x) = \sin(\pi x)$ on $[0,a]$, we have

$$a := \left[0, \frac{\sqrt{6}}{\pi}, 0, \frac{\sqrt{14}}{\pi^3}\left(\pi^2 - 15\right), 0, \frac{\sqrt{22}}{\pi^5}\left(\pi^4 - 105\pi^2 + 945\right), 0, \ldots\right]$$

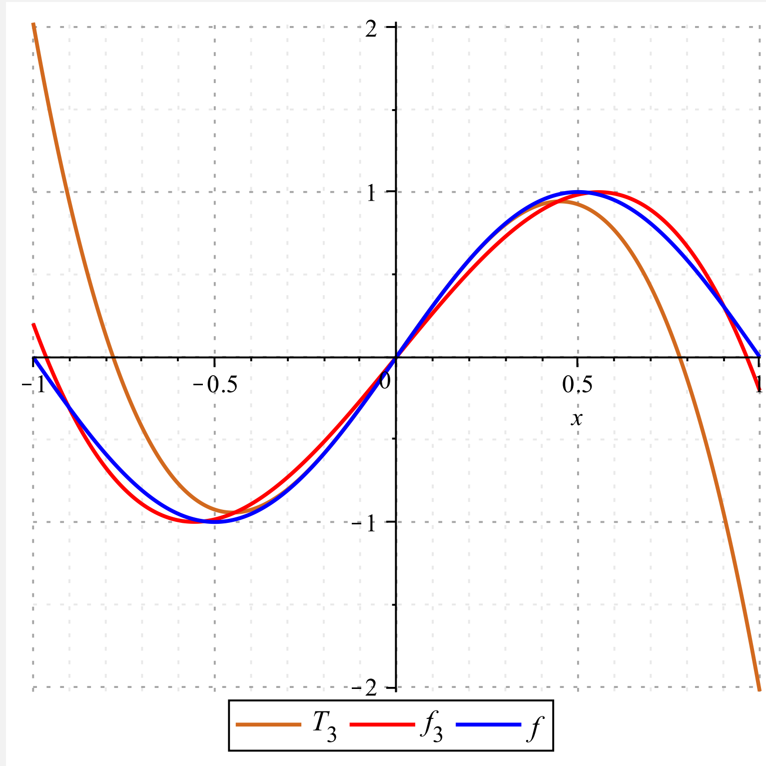
$$\sin_3(x) = \frac{\sqrt{6}}{\pi}\, p_1(x) + \frac{\sqrt{14}}{\pi^3}\left(\pi^2 - 15\right) p_3(x) = -\frac{15}{2}\,\frac{\pi^2 - 21}{\pi^3}\, x + \frac{35}{2}\,\frac{\pi^2 - 15}{\pi^3}\, x^3$$

Interlude: Legendre Expansion Graph

$f(x) = \sin(\pi x)$

$f_3(x)$: Legendre expansion

$T_3(x)$: Taylor expansion



Basic Topology of $\mathbb{R}^n$

Definition (*Total Recall:)*)

Open ball: $B(\vec{c}; r) = \{\vec{x} \mid \|\vec{x} - \vec{c}\| < r\} \subseteq \mathbb{R}^n$

Punct'd ball: $B'(\vec{c}; r) = \{\vec{x} \mid 0 < \|\vec{x} - \vec{c}\| < r\} \subset \mathbb{R}^n$; NB: $\vec{c} \notin B'(\vec{c}; r)$

Interior point: $\vec{a} \in \mathrm{int}(S)$ iff $\exists\, \varepsilon > 0$ such that $B(\vec{a}; \varepsilon) \subset S$

Open set: S is *open* iff $S = \mathrm{int}(S)$

Accum point: $\vec{a}$ in an *accumulation pt* of S iff $\forall \varepsilon > 0\ \ [B'(\vec{a}; \varepsilon) \cap S] \neq \emptyset$

Derived set: $S' = \{$all accumulation pts of $S\}$

Closed set: S is *closed* iff $S' \subseteq S$

Closure: The closure of S is $\overline{S} = S \cup S'$

Boundary pt: $\vec{b}$ is a *boundary pt* of S iff $B(\vec{b}; \varepsilon)$ contains points both of S and S complement for all $\varepsilon > 0$

Boundary: $\partial S = \{$all boundary pts of $S\}$

Isolated pt: $\vec{a}$ in an *isolated pt* of S iff $\exists\, \varepsilon > 0\ \ [B'(\vec{a}; \varepsilon) \cap S] = \emptyset$