Proper Stichens

Proposition (Open Sets)

1. *If $\mathcal{I}$ is an indexing set for a family of open sets $\{O_i\}$, then the set $\mathcal{O} = \bigcup_{i\in\mathcal{I}} O_i$ is open.* *(Arbitrary unions of open sets are open.)*
2. *If $\{O_i\}_{i=1}^n$ is a finite family of open sets, then $\mathcal{O} = \bigcap_{i=1}^n O_i$ is open.* *(Finite intersections of open sets are open.)*

Examples

1. Let $O_x = (-x, x)$ for $x \in (0,1) = \mathcal{I}$. Then
$$\bigcup_{i\in\mathcal{I}} O_i = ? \qquad \bigcap_{i\in\mathcal{I}} O_i = ?$$
2. Let $P_i = \left(-1 - \frac{1}{i}, 1 - \frac{1}{i}\right)$ for $i = 1..n$. Then
$$\bigcap_{i=1}^n P_i = ? \qquad \bigcup_{i=1}^n P_i = ?$$

Closed to Stichens

Proposition (Closed Sets)

1. *If $\mathcal{I}$ is an indexing set for a family of closed sets $\{F_i\}$, then the set $\mathcal{F} = \bigcap_{i\in\mathcal{I}} F_i$ is closed.* *(Arbitrary intersections of closed sets are closed.)*
2. *If $\{F_i\}_{i=1}^n$ is a finite family of closed sets, then $\mathcal{O} = \bigcup_{i=1}^n F_i$ is closed.* *(Finite unions of closed sets are closed.)*

Examples

1. Let $F_k = \left[-1 + \frac{1}{k}, 1 - \frac{1}{k}\right]$ for $k \in \mathbb{N}$. Then
$$\bigcap_{k\in\mathbb{N}} F_k = ? \qquad \bigcup_{k\in\mathbb{N}} F_k = ?$$
2. Let $H_i = \left[-1, 1 - \frac{1}{i}\right]$ for $i = 1..n$. Then
$$\bigcap_{i=1}^n H_i = ? \qquad \bigcup_{i=1}^n H_i = ?$$

Proper Themes

Theorem (Bolzano-Weierstrass Theorem)

A bounded, infinite subset of $\mathbb{R}^n$ has an accumulation point.

Proof.

Lion in the desert. □

Theorem (Heine-Borel Theorem)

A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Theorem (Cantor Intersection Theorem)

Let $\{F_k\}$ be a sequence of nested $(F_{k+1} \subseteq F_k)$, closed, nonempty sets for $k \in \mathbb{N}$ with F_1 being bounded. Then

$$F = \bigcap_{k=1}^{n} F_k$$

is closed and nonempty.

CIT

Proof. (Cantor Intersection Theorem).

I. If F is finite for some, then done.

II. Each F_n is infinite. Define $S = \bigcap_{k=1}^{\infty} F_k$.

1. S is closed.
2. 2.a Define the sequence $A = \{a_k\}$ by choosing distinct points $a_k \in F_k$ for each k. *Uses: F_k's are infinite.*

   2.b Since F_1 is bounded, the sequence forms a bounded, infinite set.

   2.c Therefore A has an accumulation pt a. *Bolzano-Weierstrass!*

   2.d Let $r > 0$ and set $B = B'(a; r)$. Since a is an acc pt of A, then B contains ∞ many pts of A. As the F_k's are nested, B also must contain ∞ many pts of F_k. Whence a is an acc pt of F_k.

   2.e F_k is closed, so $a \in F_k$.

   2.f The F_k are nested, so $a \in \bigcap_k F_k$; i.e., the intersection is nonempty.

□

Sample Intersections

Examples (CIT)

1. Define: $F_0 = [0, 1]$; $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] = F_0 - (\frac{1}{3}, \frac{2}{3})$;
   $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$; &c. Hence
   $$F_n = \bigcup_{k=0}^{\lfloor 3^n/2 \rfloor} \left[\frac{2k}{3^n}, \frac{2k+1}{3^n}\right]_{J(k,n)}$$
   Let $\mathcal{C} = \bigcap_n F_n$. Whence *CIT* $\implies$ $\mathcal{C}$ is nonempty and closed.

2. Let $H_n = [n, \infty)$. Then H_n is a sequence of nested, closed sets.
   But $\bigcap_n H_n = ?$

3. Set $J_n = (-\frac{n+1}{n^2}, \frac{n+1}{n^2})$. Then J_n is a sequence of bounded, nested sets.
   But $\bigcap_n J_n = ?$

Disconnection

Connected and Separated Sets

Separated: Two sets A and B are *separated* iff $A \cap \overline{B} = \emptyset = \overline{A} \cap B$.

Connected: A set S is *connected* iff S is not the union of 2 nonempty, separated sets.

Arcwise conn: Any two points in S are conn by a path inside S.

Disconnected: A set is *disconnected* iff S is not connected.

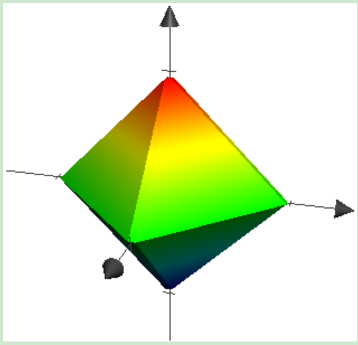
Region: A *region* is a connected set that may contain boundary points (may be neither open or closed).

Proposition

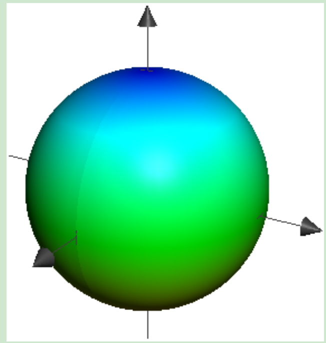
1. *Disjoint sets are separated if neither contains acc pts of the other.*
2. *Arcwise connected sets are connected*
3. *A nonempty, open, connected set is arcwise connected.*

Interlude

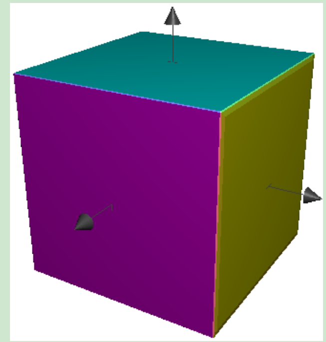
Example (Unit Balls in $\mathbb{R}^2$)



$|x| + |y| = 1$



$\sqrt{x^2 + y^2} = 1$



$\max(|x|, |y|) = 1$

Proposition

The open sets are the same under each of the metrics above.

Limits and Continuity

Definition (Limit)

- Let $f : D \to \mathbb{R}$, and let $(a, b) \in D' \subseteq \mathbb{R}^2$. Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

iff $[\forall \varepsilon > 0]\ [\exists \delta > 0]\ [\forall (x, y) \in D]$, if $\|(x, y) - (a, b)\| < \delta$, then $|f(x, y) - L)| < \varepsilon$.

- Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D' \subseteq \mathbb{R}^n$. Then

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$

iff $[\forall \varepsilon > 0]\ [\exists \delta > 0]\ [\forall \vec{x} \in D]$, if $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - L| < \varepsilon$.

- Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D' \subseteq \mathbb{R}^n$. Then

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$

iff $[\forall \varepsilon > 0]\ [\exists \delta > 0]$, $f\big(D \cap B'(\vec{a}; \delta)\big) \subseteq B(L; \varepsilon)$.

Limiting Examples

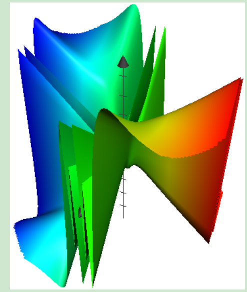
Example (*Good Function! Biscuit!*)

Let $f(x,y) = x\sin(1/y) + y\sin(1/x)$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Proof. Let $\delta(\varepsilon) = \varepsilon/2$. And

$$|f(x,y)| \leq |x| + |y|$$

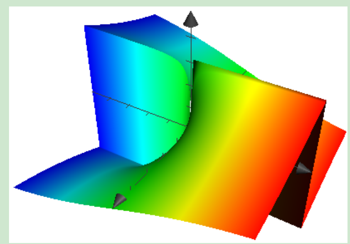


Example (*Bad Function! No biscuit!*)

Let $g(x,y) = \arctan(y/x)$. Then

$$\lim_{(x,y)\to(0,0)} g(x,y) \text{ D.N.E.}$$

Proof. Observe that $\lim_{t\to 0} g(t,t) = \pi/4$ and $\lim_{t\to 0} g(-t,t) = -\pi/4$.



Algebra of Limits

Theorem (The Algebra of Limits)

Let $f, g\colon D \to \mathbb{R}$ *and* $\vec{a} \in D'$. *Suppose* $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L_f$ *and* $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = L_g$. *Then*

1. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = L_f \pm L_g$
2. $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \cdot g(\vec{x}) = L_f \cdot L_g$
3. $\lim_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{L_f}{L_g}$ *as long as* $L_g \neq 0$
4. $\lim_{\vec{x}\to\vec{a}} |f(\vec{x})| = |L_f|$
5. *if* $f(\vec{x}) \underset{(\leq)}{<} g(\vec{x})$ *on some* $B'(\vec{a}; r)$, *then* $L_f \leq L_g$

Continuity

Definition (Continuity)

Let $f : D \to \mathbb{R}$, and $(a, b) \in D \subseteq \mathbb{R}^2$. Then f is *continuous* at (a, b) iff

- $[\forall \varepsilon > 0]$ $[\exists \delta > 0]$ $[\forall (x, y) \in D]$, if $\|(x, y) - (a, b)\| < \delta$, then $|f(x, y) - f(a, b))| < \varepsilon$.

Let $f : D \to \mathbb{R}$, and let $\vec{a} \in D \subseteq \mathbb{R}^n$. Then f is *continuous* at $\vec{a}$ iff

- $[\forall \varepsilon > 0]$ $[\exists \delta > 0]$ $[\forall \vec{x} \in D]$, if $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

- $[\forall \varepsilon > 0]$ $[\exists \delta > 0]$ $f\big(D \cap B(\vec{a}; \delta)\big) \subseteq B\big(f(\vec{a}); \varepsilon\big)$.

- $[\forall O \subseteq \mathbb{R}$, open set$]$ $f^{-1}(O) \subseteq \mathbb{R}^n$ is an open set.

Proposition

f is continuous at $\vec{a}$ iff $\big[\forall \{\vec{a}_n\}\big]$ *if $\vec{a}_n \to \vec{a}$, then* $f(\vec{a}_n) \to f(\vec{a})$

Algebra of Continuity

Theorem (The Algebra of Continuity)

Let $f, g : D \to \mathbb{R}$ be continuous at $\vec{a} \in D$. Then

1. $f \pm g$ *is continuous at* $\vec{a}$
2. $f \cdot g$ *is continuous at* $\vec{a}$
3. f/g *is continuous at $\vec{a}$ as long as* $g(\vec{a}) \neq 0$
4. $|f|$ *is continuous at* $\vec{a}$

Proof.

2. $(D \subseteq \mathbb{R}^2)$ Let $\vec{a}_n \to \vec{a}$. Since $(fg)(\vec{a}_n) = f(\vec{a}_n)\, g(\vec{a}_n)$, and f & g are continuous at $\vec{a}$, we have $f(\vec{a}_n)\, g(\vec{a}_n) \to f(\vec{a})\, g(\vec{a}) = (fg)(\vec{a})$. Thus $(fg)(\vec{a}_n) \to (fg)(\vec{a})$ for any sequence $\vec{a}_n \to \vec{a}$; hence, fg is continuous at $\vec{a}$. □

(Note: Thm 10.2.9 has problems: g & f can't be composed as $\mathrm{range}(f) \subset \mathbb{R}^1$, but $\mathrm{dom}(g) \subset \mathbb{R}^2$. So $\mathrm{range}(f) \not\subseteq \mathrm{dom}(g)$.

Continuously Reverted

Proposition

$f : \mathbb{R}^n \to \mathbb{R}$ is continuous iff

- *the preimage of any open set (in $\mathbb{R}^1$) is open (in $\mathbb{R}^n$).*
- *the preimage of any closed set (in $\mathbb{R}^1$) is closed (in $\mathbb{R}^n$).*

Proof.

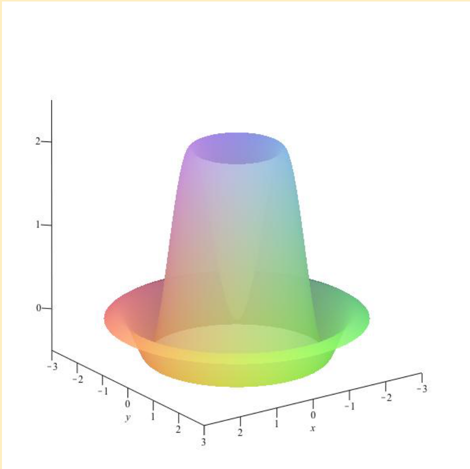
($\Rightarrow$) Assume f is cont and S is open in $\mathbb{R}^1$.
Let $\vec{a} \in f^{-1}(S)$; i.e. $f(\vec{a}) \in S$. For some $r > 0$, then $B(f(a); r) \subseteq S$.
Whence there is a $\delta > 0$, s.t. $f(B(\vec{a}; \delta)) \subseteq B(f(a); r) \subseteq S$.
Hence $B(\vec{a}; \delta) \subseteq f^{-1}(S)$.

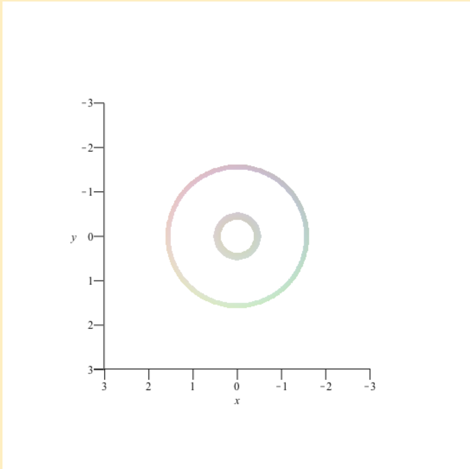
($\Leftarrow$) Assume $f^{-1}(S)$ is open whenever S is open.
Let $\vec{a} \in f^{-1}(S)$ and $\varepsilon > 0$. Thence $f^{-1}(B(f(\vec{a}; \varepsilon))$ is open.
Thus there is a $\delta > 0$ s.t. $B(\vec{a}; \delta) \subseteq f^{-1}(B(f(\vec{a}; \varepsilon))$.
Apply f to have $f(B(\vec{a}; \delta)) \subseteq B(f(\vec{a}; \varepsilon))$. □

Continuously Pictured

Preimage

Let $f(x, y) = 4\sin(x^2 + y^2)\, e^{(-(x^2+y^2)/2)}$





$$S = \left(\frac{1}{2}, 1\right) \implies f^{-1}(S) = \{0.37 < \|\vec{x}\| < 0.54\} \bigcup \{1.50 < \|\vec{x})\| < 1.78\}$$

Uniform

Definition (Uniform Continuity)

A function $f\colon D \to \mathbb{R}$ is *uniformly continuous on* D iff for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $\vec{x}_1, \vec{x}_2 \in D$, if $\|\vec{x}_1 - \vec{x}_2\| < \delta$, then $|f(\vec{x}_1) - f(\vec{x}_2)| < \varepsilon$.

Theorem

If f is continuous on D, and D is closed & bounded (compact), then

1. *f is bounded,*
2. *f attains extreme values (max and min),*
3. *f is uniformly continuous on D.*

Proof (Homework).

1. Hint: Assume not, then look at $f^{-1}(a_n)$ where $a_n \to \infty$.
2. *Bolzano-Weierstrass in action.*
3. Hint: Assume not. Create sequences $\vec{x}_n$, $\vec{y}_n$ that converge to $\vec{a}$, but have $|f(\vec{x}_n) - f(\vec{y}_n)| > \varepsilon$. Cont gives a contradiction.

□

Connecting to Rudolph Otto

Theorem

Let $f\colon D \to \mathbb{R}$ be continuous and let S be a connected subset of D. Then $f(S)$ is connected. (A connected set in $\mathbb{R}$ is an interval.)

Proof.

Suppose $f(S) = A \cup B$ with A & B nonempty, separated sets in $\mathbb{R}$. Define $G = S \cap f^{-1}(A)$ and $H = S \cap f^{-1}(B)$.

1. $S = G \cup H$ since $f\colon S \underset{\text{onto}}{\longrightarrow} f(S)$.
2. Let $\vec{y} \in A$. ($A \neq \emptyset$.) $\exists \vec{x} \in S$ s.t. $f(\vec{x}) = \vec{y}$. Thus $\vec{x} \in G \implies G \neq \emptyset$. Similarly, $H \neq \emptyset$.
3. Let $\vec{p} \in \overline{G} \cap H$. If $\vec{p} \in G$, then $\vec{p} \in G \cap H$. Then $\vec{p} \in f^{-1}(A \cap B)$; i.e., $f(\vec{p}) \in A \cap B = \emptyset$. Thus $\vec{p} \notin G$, whence $\vec{p} \in G'$ and $f(\vec{p}) \in B$. Since $\overline{A} \cap B = \emptyset$ and $\vec{p} \in B$, $\exists \varepsilon > 0$ s.t. $B(f(\vec{p}); \varepsilon) \cap A = \emptyset$. Since f is cont, $\exists \delta > 0$ s.t. $f(B(\vec{p}; \delta)) \subset B(f(\vec{p}); \varepsilon)$. Then $B(\vec{p}; \delta) \cap G$ is empty contrary to $\vec{p} \in G'$. Hence $\overline{G} \cap H = \emptyset$. Similarly $G \cap \overline{H} = \emptyset$.
4. Whereupon S is separated by G and H. *oops* $\rightarrow\leftarrow$

□

Fun with Functions

Problem (Functions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Let A and B be subsets of the domain and range of f, respectively. Then

$$f(A) = \{y \in \mathbb{R} \mid f(a) = y \text{ for some } a \in A\} \subseteq \mathrm{range}(f)$$
$$f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) = b \text{ for some } b \in B\} \subseteq \mathrm{dom}(f)$$

Give an example justifying your answer.

1. **T** or **F**: $A \subseteq f^{-1}\big(f(A)\big)$
2. **T** or **F**: $A = f^{-1}\big(f(A)\big)$
3. **T** or **F**: $A \supseteq f^{-1}\big(f(A)\big)$ *or* $f^{-1}\big(f(A)\big) \subseteq A$
4. **T** or **F**: $B \subseteq f\big(f^{-1}(B)\big)$
5. **T** or **F**: $B = f\big(f^{-1}(B)\big)$
6. **T** or **F**: $B \supseteq f\big(f^{-1}(B)\big)$ *or* $f\big(f^{-1}(B)\big) \subseteq B$

Rudolph Otto S von L

Definition (Lipschitz Condition)

If there is a constant L s.t.

$$|f(\vec{x}_1) - f(\vec{x}_2)| \leq L\,\|\vec{x}_1 - \vec{x}_1\|$$

for all $f\vec{x}_1, \vec{x}_2 \in D$, then f satisfies a *Lipschitz condition on D* (also called a "Lipschitz 1" condition).

Proposition

A function that is Lipschitz on D is uniformly continuous on D.

Proof.

Suppose f is Lipschitz with constant L.
Let $\varepsilon > 0$. Choose $0 < \delta < \varepsilon/L$. For any vectors $\vec{x}_1$ and $\vec{x}_2$ in $\mathrm{dom}(f)$ with $\|\vec{x}_1 - \vec{x}_2\| < \delta$, we have

$$|f(\vec{x}_1) - f(\vec{x}_2)| \leq L\|\vec{x}_1 - \vec{x}_2\| < L\delta < \varepsilon$$

□

Exercise

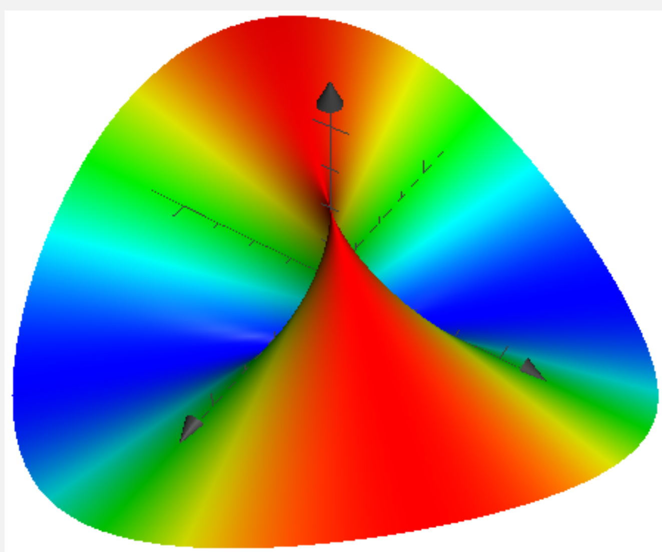
Problem (#14, pg 447)

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ *given by*

$$f(r,\theta) = \begin{cases} \frac{1}{2}\sin(2\theta) & r \neq 0 \\ 0 & r = 0 \end{cases}$$

1. *Is* f *continuous in polar coordinates?*
   Let $\theta = \pm\pi/4$*, resp., and* $r \to 0$*. Then* $\lim_{(r,\pi/4)\to\vec{0}} f(r,\theta) = 1/2$*, but* $\lim_{(r,-\pi/4)\to\vec{0}} f(r,\theta) = -1/2$*. Thus,* f *is not continuous at* $\vec{0}$ *(polar).*
2. *Write* f *in rectangular coordinates.*
   $$\tfrac{1}{2}\sin(2\theta) = \cos(\theta)\sin(\theta) = \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2}} = \frac{xy}{x^2+y^2}$$
3. *Is* f *in rectangular coordinates continuous?*
   Let $(x,y) \to \vec{0}$ *as* (t,t) *and as* $(t,-t)$*. Then* $f \to \pm 1/2$ *as* $t \to 0$*. Hence* f *is not continuous at* $\vec{0}$*.*

Exercise's Graph



$$f(r,\theta) = \begin{cases} \frac{1}{2}\sin(2\theta) & r \neq 0 \\ 0 & r = 0 \end{cases} \iff f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$$

.